Planarity of zero divisor graph of a lattice with two atoms and unique ideal

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Abstract:
Let L be a finite lattice with two atoms. In this paper we have studied zero-divisor graph of a lattice L and its unique ideal. Further explored relation between both zero divisor graphs and they are complete bipartite graph. Further given examples of planar and non planar graphs.

Keywords: Finite lattice, Zero divisor graph, Atoms, Planar .

1. INTRODUCTION

. In [3], I. Beck, introduced idea of zero divisor graph of commutative ring R with identity. He was interested in coloring of ring R. In [1], Anderson and Naseer continued this investigation. Anderson and Livingston in [2] associated a zero divisor graph to a commutative ring R with identity by different method. They defined zero divisor graph T(R) of a ring R, whose vertices are non-zero zero divisors of R and two distinct vertices x and y are adjacent if and only if xy=0.

In literature many authors have studied zero divisor graphs of rings, lattices and other graphs associated to other algebraic structures. Considering zero divisors and their annihilators, annihilating ideal graphs of different structures have studied in [9,10]. For lattices see [6,8].

In [11], Tayade explored different lattices with two atoms , studied the properties and get an idea of unique sublattice with finite number of atoms in [12]. In [13] Tayade studied zero divisor graph of a lattice with two atoms and its unique sublattice. Zero divisor graph of a finite lattice is studied in [5].

In [4], Deore and Tayade studied the zero divisor graph of a lattice and its unique ideal. Study of zero divisor graph gives idea to introduced unique sublattice .

Aim was to expose relation between graph theory and lattice theory.

In this article, zero divisor graph of a lattice L with two atoms and its unique ideal are studied. Found some are planar and some are non planar.

2. PRILIMINARIES

2.1 LATTICE THEORY

For self explaination, we recall some definitions and basic results.

A poset (L, ≤) is said to form a lattice if inf{a,b } and sup{a,b} exist in L, for any a, b∈L. In this case we write inf {a,b }=a∧b, sup{a,b}=a∨b.

A lattice with finite number of elements is finite lattice. A lattice is an algebra (L, ∧,∨) if the following conditions holds.

1)a∧a=a, a∨a=a.

2)a∧b= b∧ a, a∨b=b∨a.

3)$a\wedge(b\wedge c)= (a\wedge b)\wedge c$; $a\vee (b\vee c)= (a\vee b) \vee c$.

4)$a\vee(a\wedge b) =a$, $a\wedge(a\vee b)=a$

For a lattice L= **(**L,∧,∨), let ≤ be an order defined below.

For x, y in L, set x≤y if and only if x∧y=x, then **(**L, ≤) is an partially ordered set and any two of elements have least upper bound (lub) and greatest lower bound (glb).

Conversely, let P be an partially ordered set with glb{x, y} ,lub {x, y}∈P. Define

x∧y= glb{x, y} and x∨y= lub{x,y} then (P,∧, ∨) is lattice.

Let S be non empty subset of a lattice L. If x∧y, x∨y∈S for x, y∈S, then (S, ∧, ∨) is called sub lattice of a lattice L. For a, b ∈L, with a≤b be any two elements, the closed interval is denoted by [a,b] and [a, b] ={ x∈L: a≤ x ≤b}. The interval [a, b] is the sub lattice of L. For x∈[a,b] be any element, if their exist k∈L such that x∧k=a, x∨k=b then x is a complement of k relative to [a, b]. A lattice L is said be a bounded if there exists elements, say 0 and 1 in L such that 0≤x≤1, for all x∈L.Least element is 0 and 1 is greatest element of L. If there exist b∈L such that a∧b=0, a∨b=1, for a∈L, then b is said to be complement of a. If each element of L has complement the lattice is said to be complemented lattice.If each element of a bounded lattice has a unique complement, the lattice is said to be uniquely complemented. If b,c are elements in L such that b<x< c for no x in L then c is cover of b.

An element (0≠)a∈L is said to be atom of lattice L if a covers 0.Set of atoms of lattice L is denoted by A (L). A lattice L is said to be atomic, if for every nonzero element x∈L, their exist an atom a, such that a≤x . If a∧(b∨c)=(a∧b)∨(a∧c) for all a,b,c∈L, then the lattice L is to be distributive

For a≥ b, a∧(b∨c)=b∨(a∧c) ; a, b, c∈L, a lattice L is called modular

A non empty sub set I of L is said to be ideal of lattice L, if i∨j∈I and k∧i∈I, for i, j∈I, k∈L.

For x∈L, (x] ={y∈L:y≤x} is principal ideal generated by element x. An element a ∈L is said to be zero divisor of lattice L if their exist (0≠) b∈L such that a∧b=0. The set of all zero divisors in lattice L is denoted by Z (L). The set of non-zero zero divisors is Z* (L)={0≠k ∈L : there exist 0≠b∈L such that k∧b=0}.

In this paper all lattices are atomic, finite, with two atoms and some conditions, if not specified.

In next section we prove that there exist a finite , atomic, distributive and uniquely complemented sublattice.

2.1 GRAPH THEORY

We recall some definitions. A graph **G**=(V(G), E(G)), consist of two sets. V(G), the vertex set of graph, some times denoted by V is the set of elements called vertices. E(G) set of edges e={u,v}.

Edge is the arc or line segment joining two points u and v in set V. For two distinct vertices u and v if there exist finite number of distinct vertices $u=u_1, u_2, u_3 \ldots\ldots\ldots u_n=v$ such that each pair of vertices (u_k, u_{k+1}), $1\le k\le n-1$ is an edge then such finite sequence of vertices is called path from vertex u to v. The number of edges in the path is the length of the path. Let the length of shortest path from u to v is d(u, v). If there is no such path then d(u, v)=0. The length of largest path between two vertices in G is diameter. Cycle is the closed path and the number of edges in the cycle is its length. The length of smallest cycle in G is girth of G, denote gr(G). If G not contains cycle then gr(G)=0. A graph is complete if each pair of vertices is joined by an edge, denoted by K_n. where n is the number of vertices in graph G. For an integer r≥2, the vertex set V of graph G, is partitioned into r classes such that each edge has its end vertex in different classes, such graph is r –partite graph and is denoted by $K_{n1, n2,\ldots nr}$. where n_i is the number of vertices in i^{th} class. A undirected graph without loop and parallel edges is simple graph. For graph theory see [7].

Let L= (L, ∧, ∨) be a lattice with least element 0. An element x ∈L is said to be zero divisor if their exist a non zero element y∈L such that x∧y=0. The set of all zero divisors in lattice L is denoted by Z(L). The set of non zero zero- divisors is

Z* (L)= Z(L)\{0}. The zero divisor graph of a lattice L**,** denoted G(L) is a undirected graph whose vertex set is Z* (L) and two distinct vertices x and y are adjacent if x∧y=0.

All lattices under consideration are finite, atomic with two atoms.

3. EXISTENCE OF UNIQUE IDEAL

Let L be bounded lattice with least bound say 0 and least upper bound say 1. An element $a \in L$ is called co atom if a is covered by 1. A lattice L is said to be diamond lattice if each element $a \in L$, $a \notin \{0,1\}$ is an atom and co atom. A diamond lattice with n number of atoms is denoted by D_n.

Theorem .3.1. ([4], Theorem 2) Let L be a non diamond lattice with the least element 0 and having a finite set of atoms. Then there exists a finite, distributive and uniquely complemented ideal B of L such that A(L)=A(B).

Corollary .3.2. ([4], Corollary2) Let L be a lattice with least element 0 and has two atoms and B be unique ideal of L. Then

(a) Z (B) =B $-\{0, 1_J\}$, 1_J is the greatest element of ideal B.

(b) Z (B)$\subseteq$Z(L).

Let L be a lattice.A nonempty subset S of L said to be sublattice of L if $a \wedge b$, $a \vee b \in L$, $a, b \in S$.

Definition 3.3. Let L be a lattice and $a, b \in L$, with $a \leq b$ be any two elements. The closed interval is denoted by [a,b] and defined as $[a,b] = \{x \in L : a \leq x \leq b\}$. For an element $i \in [a,b]$, if there exist $k \in L$ such that $i \wedge k = a$ and $i \wedge k = b$ then k is complement of i relative to interval [a,b].

Theorem .3.4. Let L be a finite lattice with two atoms. Then there exists a finite, atomic, distributive and complemented ideal B of L such that A(B)=A(L).

Proof.Let A(L)={a,b} and B ={0,a,b, $a \vee b$} then obviously , B is ideal of L with A(B)= A(L). B is finite, Each element has one complement.

Uniqueness: Suppose their exist another ideal P with two atoms such that B≠P. Implies $\exists$ $x \notin B$ and $x \in P$ then x= 0 or a or b or $a \vee b$ it gives, $x \in B$ contradicts.

Corollary .3.5. Let L be a lattice and N be unique sublattice of L. Then

(a) Z^* (B) =B $-\{0, 1_J\}$, 1_J is the greatest element of sublattice N.

(b) Z (B)$\subseteq$Z(L).

Theorem.3.5. Let L be lattice with A(L)= {a,b} then |Z*(B)|=2.

Proof. Let set of atoms A(L)= {a,b}. Then $a \wedge b = 0$.

Theorem 3.6. Let (L, ≤) be a lattice with A(L)={a, b} and B be unique ideal then G(B) $\subseteq$ G(L).

Proof: The set of nonzero, zero divisors of unique ideal ; $Z^*(B) \subseteq Z^*(L)$.

G(B) has only one edge joining two atoms only, hence it lies in G(L).

Theorem 3.7. Let L be a lattice and A(L) = { a,b } be its set of atoms. Then no element of subset $[a \vee b]^u$ is element of Z(L), where $[a \vee b]^u = \{x \in L | x \geq [a \vee b\}$.

Proof: For $x \in [a \vee b]^u$, atomicity implies there exist $a \leq x$, , follows $x \wedge a = a \neq 0$.

Theorem 3.8. Let L be a Boolean lattice L= P(1,2), $\cap$, $\cup$) then L = B, is unique ideal of L.

Corrolary 3.9. . Let L be a Boolean lattice L= P(1,2), $\cap$, $\cup$) then G(B) =G(L).

A planar graph is a graph which can be redrawn in a plane.

Theorem 3.10.([7], Theorem 5.14(Kuratowski ,1930)) A graph G is planer if and only if it has no subgraph isomorphic to sudivison of K_5 or $K_{3,3}$.

Proposition 3.11. The complete bipartite graph $K_{3,3}$ is nonplaner.

Since $K_{3,3}$ is bipartite it has no odd cycle. In particular no cycle of length three. It follows that every face of a plane drawing of $K_{3,3}$, must have at least four boundary edges sum of boundary edges $b \geq 4f$, where f is the number of faces in a plane. If e is the number of edges in graph $4f \leq b \leq 2e$ i.e $2f \leq e = 9$, $f \leq 4.5$.

By Euler's formula e+2= f +n i.e 9+2=4.5+6, impossible.

Lemma .3.4. Let L be a lattice and B be its unique ideal. Then ideal I of B(sublattice) is ideal of Lattice L.

Proof. Let $B=(a\vee b]$, $a\vee b \in L$. Let I be ideal of B. Then $I=(x]$, for some $x\in B(\subseteq L)$, hence by Lemma3.3, I is ideal of L.

4.ORDER AND SIZE OF G(L)

Note: We know number of elements in vertex set is order of graph and number of edges in a graph is its size. For complete bipartite graph $K_{n,k}$, Order $(K_{n,k})=n+k$ and size $(K_{n,k})=nk$. Let L be a lattice. The elements a and b in L are non comparable we write $a\| b$. If b is cover of a then we write $a\langle b$.

Theorem 4.1. Let L be a lattice with unique ideal B.

If $A(L)=\{a,b\}$ and $a=x_1\langle x_2.....x_{k-2}\langle x_k\langle 1$ be chain and $b\| x_1, b\| x_2, ... b\| x_k$, $0\langle b\langle 1$, then

i)$Z(B)\subset Z(L)$

ii)$G(L) = K_{1,k}$

iii) Order G(L))=k+1

iv)Size G(L))=k

Proof. Let B be a unique ideal of L. Let $A(L)=\{a,b\}$ such that $a=x_1\langle x_2.....x_{k-2}\langle x_k\langle 1$, $b\| x_1, b\| x_2, ... b\| x_k$ and $0\langle b\langle 1$.

Then

i) We know $x_i\wedge x_j\neq 0$; $i\neq j$ and $x_i\wedge b=0$; for all $1\leq i\leq k$.

ii) $G(L) = K_{1,k}$

ii)Order G(L)=k+1

iii) size G(L))=1.k=k

Corollary 4.2. Let B be unique ideal of lattice L.

If $A(L)=\{a,b\}$ and $a=x_1\langle x_2.....x_{k-2}\langle x_k\langle 1$ be chain and $b\| x_1, b\| x_2, ... b\| x_k$ and $0\langle b\langle 1$,

then

i) $Z^*(B)=A(L)$

ii)$G(B) = K_{1,1}$

iii) Order G(B))=2

iv)Size G(B))=1

Theorem 4.3. Let B be a unique ideal of a lattice L.

If $A(L)=\{a,b\}$ and $a=x_1\langle x_2.....x_{k-2}\langle x_k\langle 1$, $b\| x_1, b\| x_2, ... b\| x_k$ $0\langle b\langle y_1\langle 1$, then

i) $Z^*(B)\subset Z^*(L)$

ii)$G(L) = K_{2,k}$

iii) Order G(L))=k+2

vi) size G(L))=2k

Proof. Is similar to proof of above Theorem4,1

Corollary 4.3. Let B be a unique ideal of a lattice L.

If $A(L)=\{a,b\}$ and $a=x_1\langle x_2.....x_{k-2}\langle x_k\langle 1$, $b\| x_1, b\| x_2, ... b\| x_k$ $0\langle b\langle y_1\langle 1$, then

i)$Z^*(B)=A(L)$

ii)$G(B) = K_{1,1}$

iii) Order G(B))=2

iv)Size G(B))=1

Proof.i) Let unique ideal B is proper ideal of lattice L. Let $A(L)=\{a,b\}$ such that $a\langle x_1\langle x_2.....x_{n-2}\langle x\langle 1$, $b\| x_1, b\| x_2, ... b\| x, b\langle y_1\langle y_2.....y_{k-2}\langle y\langle 1, a\| y_1, a\| y_2, ... a\| y$ then

i) $A((x])=\{a\}\neq\{b\}=A((b])$

ii) By i)

iii)OrderG(B) = 2

iii) size G(L) = 1

iv)

Theorem 4.4 . If L= (P({1,2},∩,∪) is a finite Boolean algebra then

i) Z*(L)=A(L)

ii) G(L)=G(L) = $K_{1,1}$

iii) Order G(L)=2

vi)size G(L))=1

Corollary 4.5 . If L= (P({1,2},∩,∪) is a finite Boolean algebra and B be its unique , then

i) Z*(B)= Z*(L)

ii) G(B)=G(L) = $K_{1,1}$

iii)Order G(B)=2

vi)size G(B))=1

Let L = (L, ≤) and F =(F, ≤`) be two lattices. A one one onto map g:L→F is called an lattice isomorphism if x≤y ⇔ g(x)≤`g(y) and we write it as L ≅ F

4. ZERO DIVISOR GRAPH G(L), G(B), EXAMPLES

In this section we have drawn diagram of a lattice, its zero divisor graph, and zero divisor graph of unique ideal.

Example.4.1. Let L=(D(36), |) lattice of divisors of 36 under divisibility order.

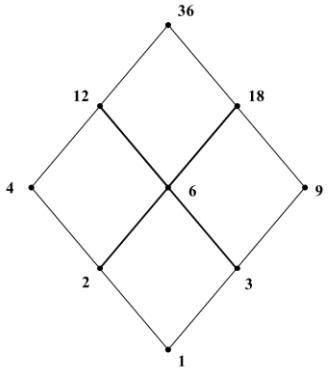

Fig1. L

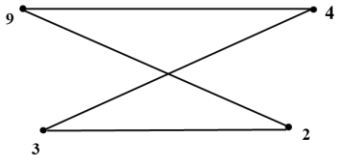

Fig 2. G(L)

Fig 3. G(N)

Example 4.2 Let: L=(D(6), |) is a lattice of divisors of 6 under divisibility order.

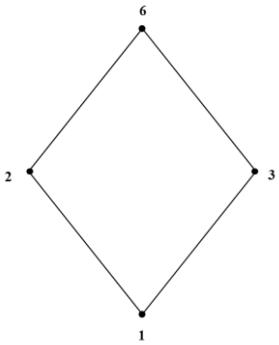

Fig 4**:** L

Fig 5: G(L)

Fig 6*:* G(B)

Example 4.3. L = (D(72), |) is lattice of divisors of 72 under divisibility order |. Then G(L)= $K_{2,3}$. And G(B)=$K_{1,1}$

5. CONCLUSIONS

Unique ideal contains two zero divisors. For lattice L with unique ideal found that G(B)⊆G(L) and former is induced subgraph of latter. In a lattice containing two atoms the zero divisor graph of a unique ideal is complete bipartite graph $K_{1,1.}$

Example 4.3, gives planar graph.

If the lattice is Boolean then it is self unique ideal.

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