Unique sub lattice of a lattice with finite number of atoms.

Author's Name
[1]P. N. Tayade,
Head, Department of Mathematics,
Lewa Educational Union's,
Dr. Annasaheb G. D. Bendale Mahila
Mahavidyalaya (Autonomous),
Jalgaon 425001,
Affiliated to KBC North Maharashtra University,
Jalgaon (M.S), India
Email ID:pntagdbmm@gmail.com

Abstract:

Let L be finite lattice with finite number of atoms. In this study we introduced a finite unique sub lattice N such that number of atoms in lattice L and in sub lattice N are same. Further formulated the number of zero divisors and number of ideals containing at most finite number of atoms. Last section figures gives a new class of lattices.

IndexTerms*: finite lattice, atom, unique sub lattice, class of lattices *(Keywords)

1. INTRODUCTION

To find the relationship between lattice theory and graph theory, associating a graph to an algebraic structure is the interest of researcher in this subject of research and has attracted considerable attention in this area in last decades. In [3],I. Beck, introduced idea of zero divisor graph of commutative ring R with identity. He defined , zero divisor graph say $T_0(R)$, of a ring R with identity, whose vertices are elements of R and two distinct vertices a and b are adjacent if and only if ab=0. He was interested in coloring of ring R. Beck conjectured that $\omega(R)=\chi(R)$,$\chi(R)$ is the chromatic number and $\omega(R)$ clique number of graph $T_0(R)$. In [1], Anderson and Naseer continued this investigation. They gave counter example of above conjecture. Anderson and Livingston in [2] associated a zero divisor graph to a commutative ring R with identity by different method. They defined zero divisor graph (R) of a ring R with identity, whose vertices are non-zero zero divisors of R and two distinct vertices a and b are adjacent if and only if ab=0.

In literature many authors have studied zero divisor graphs of rings, lattices and other graphs associated to other algebraic structures. Considering zero divisors and their annihilators, annihilating ideal graphs of different structures have studied in [8, 9]. For lattices see [5, 6].

In [10] Tayade explored different lattices with two atoms and studied the properties.

In [4] Deore and Tayade studied the zero divisor graphs of lattices and its unique ideals. Study of zero divisor graph gives idea to introduce a unique ideal related to number of atoms in a lattice.

Aim was to expose relation between graph theory and lattice theory.

In this article, finite lattice L with finite number of atoms is studied and a unique sub lattice say N was introduced. This sub lattice is finite, atomic and distributive irrespective of lattice L. Further introduced a class of such lattices. The class contains distributive, non-distributive, modular, non-modular lattices.

In this paper all lattices are finite, atomic, with finite number of atoms and some conditions, if not specified.

2 PRILIMINARIES IN LATTICE THEORY

We recall some definitions and basic results.

A poset $(L, \leq)$ is said to form a lattice if $\inf\{a,b\}$ and $\sup\{a,b\}$ exists in L, for any $a, b \in L$. In this case we write $\inf\{a,b\} = a \wedge b$, $\sup\{a,b\} = a \vee b$.

A lattice with finite number of elements is finite lattice. A lattice is an algebra (L, $\wedge,\vee$) if the following conditions holds.

1) $a\wedge a=a$, $a\vee a=a$.

2) $a\wedge b= b\wedge a$, $a\vee b=b\vee a$.

3) $a\wedge(b\wedge c) = (a\wedge b)\wedge c$; $a\vee (b\vee c) = (a\vee b) \vee c$.

4) $a\vee(a\wedge b) =a$, $a\wedge(a\vee b) =a$

Let **(**L,$\wedge,\vee$) be a lattice. Define an order $\leq$ as below.

For x, y in L, set $x\leq y$ if and only if $x\wedge y=x$, then **(**L, $\leq$) is an partially ordered set and any two elements have least upper bound (lub) and greatest lower bound (glb).

Conversely, let P be an partially ordered set with glb{x, y} ,lub {x, y}$\in$P. Define

$x\wedge y=$ glb{x, y} and $x\vee y=$ lub{x,y} then (P,$\wedge$, $\vee$) is lattice.

Let S be non-empty subset of lattice L. Then(S,$\wedge,\vee$) is called sub lattice of lattice L, if $x\wedge y$, $x\vee y\in S$ for x, y$\in$S.Let L be a lattice and a, b $\in$L, with $a\leq b$ be any two elements. The closed interval is denoted by [a,b] and defined as $[a,b] =\{x\in L: a\leq x\leq b\}$. The interval [a,b] is the sub lattice of L. For i$\in$[a,b] be any element, if their exist y$\in$L such that $i\wedge y=y\wedge i=a$, $i\vee y=y\vee i=b$ then y is a complement of i relative to [a, b].A lattice L is said be a bounded if there exist elements, say 0 and 1 in L such that $0\leq x\leq 1$, for all x$\in$L. 0 is least and 1 is greatest element of L. For a$\in$L, if there exist b$\in$L such that $a\wedge b=0$, $a\vee b=1$, then b is said to be complement of a. If each element of L has complemented the lattice is said to be complemented lattice. If each element of a bounded lattice has a unique complement, the lattice is said to be uniquely complemented. If b,c are elements in L such that $b<x< c$ for no x in L then c is cover of b.

An element $(0\neq)a\in L$ is said to be atom of lattice L if a cover 0. Set of atoms of lattice L is denoted by A (L).For every nonzero element x$\in$L,if their exist an atom a, such that $a\leq x$ then the lattice L is said to be atomic. A lattice L is said to be distributive if $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$; for all a,b,c$\in$L.

A lattice L is called modular iff $a\geq b$, $a\wedge(b\vee c)=b\vee(a\wedge c)$; for all a, b, c$\in$L.

A non-empty sub set I of L is said to be ideal of lattice L, if $a\vee b\in I$ and $k\wedge a\in I$, for a, b$\in$I, k$\in$L.

Let L be a lattice. For x$\in$L, (x] ={ y$\in$L:$y\leq x$}is principal ideal generated by element x.Let A= {a_1, a_2, a_3,..., an } be non-empty subset of L then I**={**y$\in$L**:**$y\leq a_1\vee a_2\vee a_3\vee\ldots\vee a_n$**}**is ideal of lattice L.

Let L be a lattice. An element a$\in$L is said to be zero divisor if their exist $(0\neq)$ b$\in$L such that $a\wedge b=0$. The set of all zero divisors in lattice L is denoted by Z (L). The set of non-zero zero divisors is

$Z(L)^*= \{0\neq a\in L$: there exist $0\neq b\in L$ such that $a\wedge b=0\}$.

All lattices under consideration are atomic and finite with finite number of atoms.

In next section we prove there exist a finite, atomic, distributive and uniquely complemented sub lattice.

3. EXISTANCE OF UNIQUE SUBLATTICE

Definition 3.1([4]). Let L be bounded lattice with lower bound say 0 and upper bound say 1. An element a$\in$Lis called co atom if a is covered by1. A lattice L is said to be diamond lattice if each element $a\in L, a\notin\{0,1\}$ is an atom and co atom. A diamond lattice with n atoms is denoted by D_n.

Theorem .3.2. ([4], Theorem 2) Let L be a non diamond lattice with the least element 0 and having a finite set of atoms. Then there exist a finite, distributive and uniquely complemented ideal B of L such that A (L) =A (B).

Corollary .3.3. ([4], Corollary2) Let L be a lattice with least element 0, finite number of atoms and B be its unique ideal. Then

(a) Z (B) =B –{0, 1_J}, 1_J is the greatest element of ideal B.

(b) Z (B)$\subseteq$Z(L).

Theorem 3.4. Let L be a finite lattice and A (L) = { a_1, a_2, a_3,…. a_n} be its set of atoms. Then no element of subset $[\vee a_j]^u$; $1\leq j\leq n$ is element of Z(L), where $[\vee a_j]^u=\{x\in L|\ x\geq[\vee a_j\}$.

Proof: For $x\in[\vee a_j]^u$, atomicity implies there exist $a_j\leq x$, for some j, follows $x\wedge a_j = a_j\neq 0$.

.**Lemma. 3.5.** If L be a lattice and A(L)={a_1, a_2, a_3,…. a_n} then N={ $\wedge a_i, \vee a_j$: a_i, $a_j \in$ A(L),$1 \le i \le n$, $1 \le j \le n$} is a sub lattice of L.

Proof. Let L be a lattice and A(L)={a_1, a_2, a_3,…. a_n}. Let N={ $\wedge a_i, \vee a_j$: a_i, $a_j \in$ A(L),$1 \le i \le n$, $1 \le j \le n$}, then $N \neq \phi$ and for any two a_i, a_j,: $a_i \wedge a_j$, and $a_i \vee a_j$, are elements of N and 0 is the least element and $1_J = \vee a_j$, for all $1 \le i \le n$, is the greatest element of N.

Definition 3.6. Let L be a lattice and a, b $\in$L, with $a \le b$ be any two elements. The closed interval is denoted by [a,b] and defined as [a,b] ={$x \in L: a \le x \le b$}. For an element $x \in [a,b]$, if there exist $k \in L$ such that $x \wedge k = a$ and $x \wedge k = b$ then k is complement of x relative to interval [a,b].

Theorem .3.7. Let L be a non-diamond (excluding D_2) finite lattice and having a finite set of atoms. Then there exist a finite, distributive and uniquely complemented sub lattice N of L such that A(L)=A(N).

Proof. Let L be a non-diamond (excluding D_2) bounded lattice and having a finite set of atoms A(L)= {a_1, a_2, a_3,. a_n} and by (Theorem .3.2.([4]) , Lemma 3.5, $(N, \wedge, \vee)$ is sub lattice of L.

Since A(L) is finite N is also finite. For any $x \in N$, $a_i \le x = a_i \vee a_j$ for some i and j, shows that N is atomic. We observe that, in N, every element in interval has one relative complement. It follows N is distributive.

Uniqueness. Suppose their exist another sub lattice P containing same n number atoms. Such that $N \neq P$, implies $x \notin N$ and $x \in P$ then $x = a_1 \vee a_{2.....} \vee a_{j-1} \vee a_j$ by definition of N, $x \in N$ contradicts.

Corollary 3.8. Let L be lattice then

i)$Z^*(N) = N - \{0, 1_J\}$ ii) $Z(N) \subseteq Z(L)$

Proof. i) For $x \in Z^*(N)$, $x = a_1 \vee a_{2.....} \vee a_{j-1} \vee a_{j+1} \vee \ldots \vee a_n$ and $x \wedge a_j = 0$

ii) For $x \in Z(N)$, $x = a_1 \vee a_{2.....} \vee a_{j-1} \vee a_{j+1} \vee \ldots \vee a_n$ and $x \wedge a_j = 0$ and $x = 0$ then $x \in Z(L)$.

Theorem.3.9. Let L be lattice with A(L)= {a_1, a_2, a_3,. a_n} then i)$|Z^*(N)| = 2^n - 2$, $n \ge 2$.

Proof. i) For all i, a_i is zero divisor of L. Its contribution is ${}^nC_1$, $a_j \vee a_k$ is again zero divisor, its contribution ${}^nC_2$, … continuing…. $a_1 \vee a_{2.....} \ldots \vee a_{n-1}$ is zero divisor its contribution is ${}^nC_{n-1}$, the total contribution is $2^n - 2$.

We know that every ideal is a sub lattice,next theorem gives ideals of sub lattice(ideal).

Theorem.3.10. Let L be lattice. If N is ideal of L then each ideal of N is ideal of L.

Corollary3.11. Leta lattice L with |A(L)|= n and uniqueidealN. Then N has 2^n number of ideals contained in N, havingat most n number of atoms.

Proof. Let A(L)= {a_1, a_2, a_3,…. a_n}.A ideal with no atom is one in number.Theorem .3.2.([4], Theorem 2),Number of ideals containing one atom is ${}^nC_1$.Number of ideals containing two atoms is ${}^nC_2$, …Number of ideals containing three atoms is ${}^nC_3$, continuing….Number of ideals containing n number of atoms is ${}^nC_n$,

thus L has 2^n number of ideals in N, containing at most n number of atoms.

Theorem3.12. Let L be lattice and A(L)= {a_1, a_2, a_3,…. a_n} and N unique sub lattice of L. Then

i) If 1_J and 1 are least elements of N and L respectively then i) $1_J \le x \le 1$, has no complement in L.

ii) If $y < 1_J$ then y has a complement in N and not in L.

Example 3.13. Let L be a Boolean lattice L=P(1,2,..n),$\cap$,$\cup$) then N = Lis unique sub lattice of L.

4.SOME LATTICES AND EXAMPLES

Example.4.1 Figure. fig 1is lattice and fig 2 is unique sub lattice.

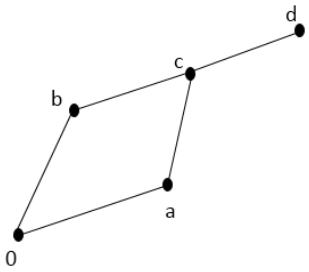

fig 1

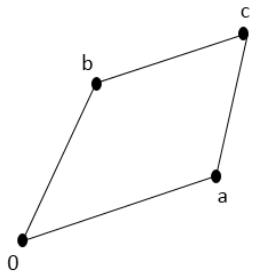

fig 2

Example4.2. Figure,fig3 is latticeL= (D(60), |) of divisors of 60 under divisibility order | and fig 4 is its unique sub lattice (D(30), |) of divisors of 30 under divisibility order |.

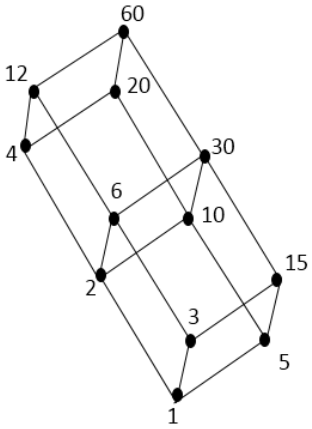

fig 3

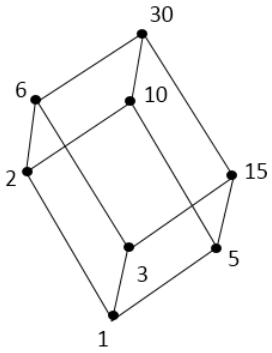

fig 4

Example4.3. Figure fig5 is diamond lattice D_3= (D_3, ≤) and fig 6is its sub lattice (D_2, ≤). Diamond lattice D_3 has two type of D_2sub lattices containing exactly two atoms and these are isomorphic to each others.

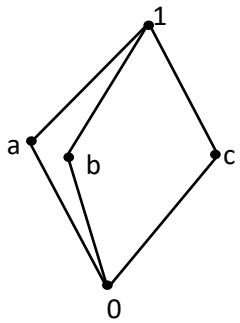

fig5

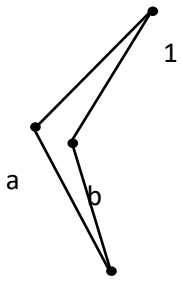

fig6

5.CONCLU

Investigated that a finite lattice containing n number of atoms contains a finite, distributive and atomic unique sub lattice. The uniquesub lattice has 2^n-2 zero divisors.Existence of lattice having 2^n number of ideals containing at most n atoms. If the latticeis Boolean then it is self-uniquesub lattice.

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