Extended (k_1, k_2)-centralizers on a C^*-algebra and some related theorems

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Abstract— In this study some fixed point theorems are established on the extended (k_1, k_2)-centralizers on C^*-algebras. Considering different conditions on the extended centralizer f such as asymptotic regularity property, asymptotically non-expansive property etc.are used to get the fixed point on a C^*-algebra $\mathcal{A}$. Also, different contractive type conditions are taken on the mapping k_2 to get some more results of fixed point for the extended centralizer f on a C^*-algebra $\mathcal{A}$. Some examples are given in support of the theorems.

Keywords: C^*-algebra, Fixed Point Theorem, Extended Centralizer

1 Introduction

The concept of C^*-algebra can be found in [5] ,[14], [3] and [11]. Because of its tremendous application in various fields of mathematics and physics C^*-algebra theory is developed quite rapidly since its introduction. Some application of C^*-algebra are found in topology, quantum mechanics,representation theory of locally compact groups, partial differential equations, numerical analysis etc.

The study of fixed point theory is also of great importance in now a days due to its wide applicability in different fields like mathematics, economics, commerce, physics, life science, computer science, engineering etc. After the well known banach contraction principle, there is an interesting development of this theory considering different spaces such as C^*-algebra valued metric spaces, b-metric spaces, S-metric space etc. In 2011, Dhompongsa et al. [6] studied some fixed point properties of C^*-algebras.

Many fixed point theorems on various mappings are also studied in C^*-algebra. Das et al., in [15], studied common fixed point results in C^*-algebra valued modular metric spaces. In 2023, Rezaei Aderyani et al. [13] worked on C^*-algebra and fixed ponit. In [16], Sarma et al. introduced the extended (k_1, k_2)-centralizer f, on a C^*-algebra $\mathcal{A}$.

In this paper, we investigate the existence of fixed point for the extended left (k_1, k_2)-centralizer f defined on the C^*-algebra $\mathcal{A}$ by considering different conditions to the self mapping k_2 .

2 Preliminaries

Here are some basic definitions which are used in finding our results:

Definition 2.1 *Let $\mathcal{A}$ be a C^*-algebra. For two mappings $k_1: \mathcal{A} \to [0, \infty)$ and $k_2: \mathcal{A} \to \mathcal{A}$ (where $k_2(e) = 0$,(the zero element of $\mathcal{A}$), if $\mathcal{A}$ is unital with unit element e), an additive self mapping f on $\mathcal{A}$ is called an extended left(right) (k_1, k_2)-centralizer if it satisfies*

$$f(xy) = f(x)y + k_1(x)f(x)k_2(y) \quad (f(xy) = xf(y) + k_1(y)f(y)k_2(x))$$

for each $x, y \in \mathcal{A}$.

f is said to be an extended (k_1, k_2)-centralizer if it is both extended left and right (k_1, k_2)-centralizer, i.e.,

$$f(xy) = f(x)y + k_1(x)f(x)k_2(y) = xf(y) + k_1(y)f(y)k_2(x) \text{ for each } x, y \in \mathcal{A}.$$

If $k_1(x) = c,\ \forall x \in \mathcal{A}$, we call f as an extended (c, k_2)-centralizer.

Remark: The class of extended (k_1, k_2)-centralizers contains the class of centralizers for $k_1 = 0$ or $k_2 = 0$.

Definition 2.2 *[1] Let K be a closed and convex subset of a Banach space $\mathcal{A}$. A self mapping T on K is called weakly contractive if, for each $x, y \in K$,*

$$\| Tx - Ty \| \leq \| x - y \| - \psi(\| x - y \|)$$

where $\psi: \mathcal{R}_+ \to \mathcal{R}_+$ is a continuous and non-decreasing such that ψ is positive on $[0, \infty)$ and $T(0) = 0, \lim t \to +\infty\, \psi(t) = \infty$

Definition 2.3 *Let C be a non-empty suset of a Banach space $(X, \|.\|)$. Then the self mapping $T: C \to C$ is called the non-expansive mapping if*

$$\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C.$$

T is said to be nonexpansive mapping with center 0 if $\| Tx \| \leq \| x \|$ for all $x \in C$.

Definition 2.4 *A self-mapping T on a normed space $(X, \|.\|)$ is said to be asymptotically regular at a point x in X if* $\| T^n x - T^{(n+1)} x \| = 0$ *as* $n \to \infty$, *where* $T^n x$ *denotes the* n^{th} *iterate of* T *at* $x \in X$.

Definition 2.5 *Let C be a non-empty suset of a Banach space $(X, \|.\|)$ and let $T: C \to C$ be a self-mapping. Then T is said to be asymptotically nonexpansive if*

$$\lim_{n\to+\infty} \sup \| T^n x - T^n y \| \leq \| x - y \| \ \ for\ all\ x, y \in C$$

Remark Every nonexpansive mapping is asymptotically nonexpansive while the converse is not true .

Definition 2.6 *Let C be a non-empty subset of real Banach space X and f a mapping from C to C. f is called mean non-expansive if for each $x, y \in C$,*

$$\| f(x) - f(y) \| \leq a \| x - y \| + b \| x - f(y) \|,\ \ a, b \geq 0, a + b \leq 1.$$

Remark Let (X, d) be a complete metric space and f be a self mapping on X.

1.(Kannan [10]) There exists a number k, $0 < k < \frac{1}{2}$, such that for each $x, y \in X$,

$$d(f(x), f(y)) \leqslant k[d(x, f(x)) + d(y, f(y))].$$

2.(Bianchini [4]) There exists a number k, $0 \leqslant k < 1$, such that for each $x, y \in X$,

$$d(f(x), f(y)) \leqslant k\max[d(x, f(x)), d(y, f(y))].$$

3.((Reich [12])) There exist non negative numbers $a,\ b,\ c$, such that $a + b + c < 1$, and for each $x, y \in X$,

$$d(f(x), f(y)) \leqslant ad(x, f(x)) + bd(y, f(y)) + cd(x, y).$$

3 Main Results

Considering k_2 as a weak contraction mapping following fixed point theorem of f is found.

Theorem 3.1 *For a unital C^*-algebra $\mathcal{A}$, let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer which satisfies the conditions :*

1. $\| f(e) \| < \frac{1}{\lambda}$, for some $\lambda > 2$ and
2. $\| k_1(e) \| \leq 1$.

Let $k_2: \mathcal{A} \to \mathcal{A}$ be a weak contraction mapping such that

$$\| k_2(x) - k_2(y) \| \leq \| x - y \| - T(\| x - y \|)$$

where $T: \mathcal{R}_+ \to \mathcal{R}_+$ is continuous and non-decreasing such that $T(0) = 0$. Then f has a fixed point.

Proof: For any elements x and y in the C^*-algebra $\mathcal{A}$,

$$\| f(x) - f(y) \| = \| f(e)x + k_1(e)f(e)k_2(x) - f(e)y - k_1(e)f(e)k_2(y) \|$$
$$\leq \| f(e) \| \| x - y \| + \| k_1(e) \| \| f(e) \| \| k_2(x) - k_2(y) \|$$
$$< \frac{1}{\lambda} + \frac{1}{\lambda} \| k_2(x) - k_2(y) \|$$
$$\leq \frac{1}{\lambda} \| x - y \| + \frac{1}{\lambda}\{\| x - y \| - T(\| x - y \|)\}$$
$$= \frac{2}{\lambda} \| x - y \| - \frac{1}{\lambda} T(\| x - y \|)$$

Now we take $x_{n+1} = fx_n$. Then the above expression becomes

$$\| x_{n+1} - x_n \| = \| fx_n - fx_{n-1} \|$$
$$\leq \frac{2}{\lambda} \| x_n - x_{n-1} \| - \frac{1}{\lambda} T(\| x_n - x_{n-1} \|)$$
$$\leq \frac{2}{\lambda} \| x_n - x_{n-1} \|$$

Thus we can see that

$$\| x_{n+1} - x_n \| \leq \frac{2}{\lambda} \| x_n - x_{n-1} \| \leq (\frac{2}{\lambda})^2 \| x_{n-1} - x_{n-2} \| \leq \ldots\ldots \leq (\frac{2}{\lambda})^n \| x_1 - x_0 \|$$

For $n \to \infty$, $\| x_{n+1} - x_n \| \to 0$, which implies that $\{x_n\}$ is a Cauchy sequence.

As $\mathcal{A}$ is complete, hence $\{x_n\}$ converges to a number x in $\mathcal{A}$.

Now,

$$\| x - fx \| = \| x - x_{n+1} + x_{n+1} - fx \|$$
$$\leq \| x - x_{n+1} \| + \| fx_n - fx \|$$
$$\leq \| \mathrm{x} - \mathrm{x}_{n+1} \| + \frac{2}{\lambda} \| x_n - x \| - \frac{1}{\lambda} T(\| x_n - x \|)$$

For $n \to \infty$, $\| \mathrm{x} - fx \| \leq 0 + \frac{2}{\lambda}.0 - \frac{1}{\lambda}.\mathrm{T}(0) = 0$ which implies that $fx = \mathrm{x}$ i.e. f has a fixed point.

Example 3.2 We define a mapping $f: \mathcal{R} \to \mathcal{R}$ such that $f(x) = \frac{x}{4}$ and $k_1: \mathcal{R} \to [0, \infty)$ where $k_1(x) = 0$ and $k_2: \mathcal{R} \to \mathcal{R}$ such that $k_2(x) = \frac{x}{3}$. Then f is an extended (k_1, k_2)-centralizer. Taking $\lambda = 3$ we have $\| f(1) \| = \frac{1}{4} < \frac{1}{3}$ and $\| k_1(1) \| = 0 < 1$. Also, $\| k_2(x) - k_2(y) \| = \| \frac{x}{3} - \frac{y}{3} \| < \frac{1}{2} \| x - y \| = \| x - y \| - \frac{1}{2} \| x - y \| = \| x - y \| - T(\| x - y \|)$ where $T: \mathcal{R}_+ \to \mathcal{R}_+$ such that $T(x) = \frac{x}{2}$ is continuous and non-decreasing such that $T(0) = 0$ which shows that k_2 is weak- contraction mapping. So all the conditions of Theorem 3.1 are satisfied and hence f has a fixed point.

In the following result, the asymptotic regularity property of f is used to get a fixed point on $\mathcal{A}$.

Theorem 3.3: For a unital C^*-algebra , let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer such that $\| k_2 \| \leq 1$ and $0 \in Ker\, f$. If $a = \| f(e) \| < 1$ and $b = \| k_1(e) \| < \frac{1}{a} - 1$, then f has a unique fixed point in $\mathcal{A}$.

Proof. For any elements x and y in the C^*-algebra , it is clear that

$$\| f(x) - f(y) \| = \| f(ex) - f(ey) \|$$
$$\leq \| f(e) \| [\| x - y \| + \| k_1(e) \| \| k_2(x) - k_2(y) \|] \quad (1)$$

Applying the given conditions it is found that (3.1) is equal to

$$\| f(x) - f(y) \| \leq a[\| x - y \| + b \| k_2(x) - k_2(y) \|]$$
$$\leq a[\| x \| + \| y \| + b(\| k_2 \| \| x \| + \| k_2 \| \| y \|)]$$
$$= a[(\| x \| + \| y \|) + b \| k_2 \| (\| x \| + \| y \|)]$$
$$\leq a(1 + b)(\| x \| + \| y \|)$$
$$= c(\| x \| + \| y \|),$$

where $c = a(1 + b) < 1$ since $b < \frac{1}{a} - 1$.

Now, $f(0) = 0$, So $\| f(x) - f(0) \| \leq c \| x \|$ i.e., $\| f(x) \| \leq c \| x \|$ and hence, $\| f^m(x) \| \leq c^m \| x \|$ for $m \in \mathbb{N}$

Since $c < 1$, so, it can be shown that f is asymptotically regular at every point x of $\mathcal{A}$.

Now, proceeding as the Theorem 2.5 of [7] we conclude that f has a unique fixed point in $\mathcal{A}$.

Example 3.4: Let $f: l^1 \to l^1$ be the extended (k_1, k_2)-centralizer defined by $f(\{x_1, x_2, \dots\}) = \{\frac{x_1}{3}, \frac{x_2}{3}, 0,0 \dots\}$. Also $k_1: l^1 \to [0, \infty)$ and $k_2: l^1 \to l^1$ be such that $k_1(\{x_1, x_2, \dots\}) = \frac{1}{4}$ and $k_2(\{x_1, x_2, \dots\}) = \{0,0,0,0..\}$. Clearly $\| k_2 \| < 1$ and $0 \in Ker f$. Also, $a = \| f(\{1,1, \dots\}) \| = \| \{\frac{1}{3}, \frac{1}{3}, 0,0,..\} \| = |\frac{1}{3}| + |\frac{1}{3}| + 0 + 0 + \dots = \frac{2}{3} < 1$ and $b = \| k_1(\{1,1, \dots\}) \| = \frac{1}{4} < \frac{3}{2} - 1$. So, all the conditions of the Theorem 3.2 are satisfied. Hence, f has a unique fixed point in l^1, which is $\{0,0, \dots .\}$ here.

Considering the k_2 mapping as a nonexpansive mapping, the following result is established.

Theorem 3.5: For a unital C^*-algebra $\mathcal{A}$, let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer which satisfies the conditions:

1. $0 \in Ker\, f$ and $\| f(e) \| \leq \lambda \in (0,1]$, $k_1(e) = 1$,
2. $k_2 : \to$ is nonexpansive,
3. for some $p < \frac{1}{3\lambda}$, $\| f(x) - f(y) \| \leq p(\| f(x) - f(e)k_2(y) \| + \| f(y) - f(e)k_2(x) \|)$, $for\ all\ x, y \in \mathcal{A}$

Then f has a unique fixed point in $\mathcal{A}$.

Proof. For any elements x and y in the C^*-algebra $\mathcal{A}$,

$$\| f(x) - f(y) \| \leq p(\| f(x) - f(e)k_2(y) \| + \| f(y) - f(e)k_2(x) \|)$$
$$= p(\| f(e)x + k_1(e)f(e)k_2(x) - f(e)k_2(y) \| + \| f(e)y + k_1(e)f(e)k_2(y) - f(e)k_2(x) \|)$$

After applying condition (i) the above expression becomes

$$\| f(x) - f(y) \| \leq p(\| f(e)(x + k_2(x) - k_2(y)) \| + \| f(e)(y + k_2(y) - k_2(x)) \|),$$
$$\leq p \| f(e) \| (\| x + k_2(x) - k_2(y) \| + \| y + k_2(y) - k_2(x) \|)$$
$$\leq p\lambda(\| x + k_2(x) - k_2(y) \| + \| y + k_2(y) - k_2(x) \|) \text{ bycondition(i)}$$
$$\leq p\lambda (\| x \| + \| x - y \| + \| y \| + \| y - x \|), \text{ bycondition(ii)}$$
$$\leq p\lambda (\| x \| + \| x \| + \| y \| + \| y \| + \| y \| + \| x \|)$$
$$= 3p\lambda (\| x \| + \| y \|)$$
$$= c(\| x \| + \| y \|), \text{ where } c = 3p\lambda < 1, \text{ since } p < \frac{1}{3\lambda}$$

Here it is observed that the last expression is similar as in Theorem 3.2. So proceeding as in that manner it is found that f has a unique fixed point in $\mathcal{A}$.

Remark: The conditions (ii) and (iii) of the above result can be replaced by the following:

$$\| f(x) - f(y) \| \leq \| f(x) - f(e)k_2(x) \| + \| f(y) - f(e)k_2(y) \| \quad \text{forall } x, y \in \mathcal{A}.$$

[By the above condition,] for $x, y \in \mathcal{A}$,

$$\| f(x) - f(y) \| \leq \| f(x) - f(e)k_2(x) \| + \| f(y) - f(e)k_2(y) \|$$
$$= \| f(e)x + k_1(e)f(e)k_2(x) - f(e)k_2(x) \| + \| f(e)y + k_1(e)f(e)k_2(y) - f(e)k_2(y) \|$$
$$= \| f(e)x + f(e)k_2(x) - f(e)k_2(x) \| + \| f(e)y + f(e)k_2(y) - f(e)k_2(y) \| \text{ (since} k_1(e) = 1)$$
$$\leq \| f(e) \| (\| x \| + \| y \|)$$
$$\leq \lambda(\| x \| + \| y \|), \text{ where } \lambda < 1.$$

Now, as in Theorem 3.2, f has a unique fixed point in $\mathcal{A}$.

In a similar way, the following result is obtained.

Theorem 3.6: For a unital C^*-algebra $\mathcal{A}$, let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer satisfying the conditions:

1. $0 \in Ker\, f$ and $\| f(e) \| \leq \lambda \in (0,1]$,
2. $k_1(e) = 1$ and $\| k_2 \| < 1$,
3. $\| f(x) - f(y) \| \leq \| f(x) - f(e)x \| + \| f(y) - f(e)y \|$ for all $x, y \in \mathcal{A}$.

Then f has a unique fixed point in $\mathcal{A}$.

In the following Theorem, asymptotically regular property is used and k_2 is considered as mean nonexpansive mapping.

Theorem 3.7: Let f be an extended (k_1, k_2)-centralizer on a unital C^*-algebra $\mathcal{A}$ and $k_2: \mathcal{A} \to \mathcal{A}$ be a mean nonexpansive mapping i.e.

$$\| k_2(x) - k_2(y) \| \le a \| x - y \| + b \| x - k_2(y) \|, a + b \le 1, a, b \ge 0.$$

If f satisfies the conditions: i) $\| k_1(e) \| = 1$, and ii) $\| f(e) \| < \frac{1}{\lambda}, \lambda > 2, k_2(0) = 0$, then f has a fixed point.

Proof: For any elements x and y in the C^*-algebra ,

$$\| f(x) - f(y) \| = \| f(e)x + k_1(e)f(e)k_2(x) - f(e)y - k_1(e)f(e)k_2(y) \|$$
$$\le \| f(e) \| \| x - y \| + \| k_1(e) \| \| f(e) \| \| k_2(x) - k_2(y) \|$$
$$< \frac{1}{\lambda} \| x - y \| + \frac{1}{\lambda} \| k_2(x) - k_2(y) \|$$
$$\le \frac{1}{\lambda} \| x - y \| + \frac{1}{\lambda} \{a \| x - y \| + b \| x - k_2(y) \|\}$$
$$= (\frac{1+a}{\lambda}) \| x - y \| + \frac{b}{\lambda} \| x - k_2(y) \|$$
$$\le (\frac{1+a}{\lambda})(\| x \| + \| y \|) + \frac{b}{\lambda}(\| x \| + \| k_2(y) \|)$$
$$= (\frac{1+a+b}{\lambda}) \| x \| + (\frac{1+a}{\lambda}) \| y \| + \frac{b}{\lambda} \| k_2(y) \|.$$

For $x = 0, f(0) = f(e).0 + k_1(e)f(e)k_2(0) = 0$as by assumption $k_2(0) = 0$.

Therefore,

$$\| f(x) - f(0) \| \le (\frac{1+a+b}{\lambda}) \| x \| + 0$$
$$= (\frac{1+a+b}{\lambda}) \| x \|$$
$$= c \| x \|, \text{ where } \frac{1+a+b}{\lambda} = c < 1$$
$$\Rightarrow \| fx \| \le c \| x \|$$
$$\Rightarrow \| f^m x \| \le c^m \| x \|$$

Now for m in $\mathbb{N}$,

$$\| f^{m+1}x - f^m x \| = \| f(f^m x) - f(f^{m-1}x) \|$$
$$\le c \| f^m x \| + (\frac{1+a}{\lambda}) \| f^{m-1}x \| + \frac{b}{\lambda} \| k_2(f^{m-1}x) \|$$
$$\le cc^m \| x \| + (\frac{1+a}{\lambda})c^{m-1} \| x \| + \frac{b}{\lambda} \| k_2(c^{m-1} \| x \|) \|$$
$$= c^{m+1} \| x \| + (\frac{1+a}{\lambda})c^{m-1} \| x \| + \frac{b}{\lambda} \| k_2(c^{m-1} \| x \|) \| \quad (2)$$

As $c < 1$ and $k_2(0) = 0$, if $m \to \infty$ then the equation (1) will reduced to 0, which implies that f is asymptotically regular at every point x in $\mathcal{A}$.

Now, proceeding as Theorem 2.5 of [7] it can be shown that f has a unique fixed point.

The following result is based on asymptotically non-expansive mapping.

Theorem 3.8: For a unital C^*-algebra $\mathcal{A}$, let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer which satisfies the conditions :

1. $\| f(e) \| \le 1$, and
2. $\| k_1(e) \| \le \frac{1}{\lambda^m}$ for some natural number m and $\lambda > 1$ and $k_2(0) = 0$.

If f is asymptotically non-expansive then f has a fixed point.

Proof: For any elements x and y in the C^*-algebra $\mathcal{A}$,

$$\| f(x) - f(y) \| = \| f(e)x + k_1(e)f(e)k_2(x) - f(e)y - k_1(e)f(e)k_2(y) \|$$
$$\le \| f(e) \| \| x - y \| + \| k_1(e) \| \| f(e) \| \| k_2(x) - k_2(y) \|$$
$$\le \| x - y \| + \frac{1}{\lambda^m} \| k_2(x) - k_2(y) \|$$

For natural number m,

$$\| f^m(x) - f^m(y) \| = \| f(f^{m-1}(x)) - f(f^{m-1}(y)) \|$$
$$\le \| f^{m-1}(x) - f^{m-1}(y) \| + \frac{1}{\lambda^m} \| k_2(f^{m-1}(x)) - k_2(f^{m-1}(y)) \|$$
$$\le \| f^{m-2}(x) - f^{m-2}(y) \| + \frac{1}{\lambda^m} \| k_2(f^{m-2}(x)) - k_2(f^{m-2}(y)) \| + \frac{1}{\lambda^m} \| k_2(f^{m-1}(x)) - k_2(f^{m-1}(y)) \|$$
$$\ldots\ldots\ldots$$
$$\le \| x - y \| + \frac{1}{\lambda^m} \| k_2(x) - k_2(y) \| + \ldots + \frac{1}{\lambda^m} \| k_2(f^{m-2}(x)) - k_2(f^{m-2}(y)) \| + \frac{1}{\lambda^m} \| k_2(f^{m-1}(x)) - k_2(f^{m-1}(y)) \|$$

If $m \to \infty$ then $\frac{1}{\lambda^m} \to 0$,

which leads to $\limsup_{m\to\infty} \| f^m(x) - f^m(y) \| \le \| x - y \|$, showing that f is asymptotically non-expansive mapping. Also, if f is asymptotically non-expansive mapping with center 0, then by the Theorem 3.2 of [2] f has a fixed point 0.

In the next part, various well established results are used to find fixed point for the the extended centralizer f on a C^*-algebra $\mathcal{A}$. Banach contraction principle is used in the following theorem to obtained a fixed point.

Theorem 3.9: For a unital C^*-algebra $\mathcal{A}$, let $f: \mathcal{A} \to \mathcal{A}$ be an extended (k_1, k_2)-centralizer which satisfies the conditions :

1. $\| f(e) \| \le \frac{1}{\lambda}$, for some $\lambda > 1$ and
2. $\| k_1(e) \| < \lambda - 1, e$ being the unit element of .

If $k_2: \mathcal{A} \to \mathcal{A}$ is nonexpansive then f has a unique fixed point in $\mathcal{A}$.

Proof: For any elements x and y in the C^*-algebra ,

$$\| f(x) - f(y) \| = \| f(ex) - f(ey) \|$$

$=\| f(e)x+k_1(e)f(e)k_2(x)-f(e)y-k_1(e)f(e)k_2(y) \|$
$\leq \| f(e) \| \| x-y \| + \| k_1(e) \| \| f(e) \| \| k_2(x)-k_2(y) \|$
$\leq \| f(e) \| \| x-y \| + \| k_1(e) \| \| f(e) \| \| x-y \|$, ($k_2$ being nonexpansive)

Applying the given condition we get

$\| f(x)-f(y) \| \leq \frac{1}{\lambda}(\| x-y \| + \| k_1(e) \| \| x-y \|)$
$=\frac{1}{\lambda}(1+\| k_1(e) \|) \| x-y \|$
$=p \| x-y \|$, where $p=\frac{1}{\lambda}(1+\| k_1(e) \|)<1$

Thus, f is a contraction on the C^*-algebra , and hence by Banach's contraction principle, f has a unique fixed point in $\mathcal{A}$.

The following result is based on Kannan condition for fixed point Theorem (refer to [10]).

Theorem 3.10: On the unital C^*-algebra $\mathcal{A}$, let f be an extended (k_1,k_2)-centralizer satisfying:

$$\| f(x)-f(y) \| \leq \left(\| \frac{1}{p}k_2(x) \| + \| \frac{1}{p}k_2(y) \|\right) \text{ forall } x,y\in,$$

and for some $p=3q, q\in\mathbb{Z}^+$. Then f has a unique fixed point if $f(e)=1$ and $k_1(e)=\frac{1}{q}$.

Proof: For any elements x and y in the C^*-algebra $\mathcal{A}$,

$\| f(x)-f(y) \| \leq (\| \frac{1}{p}k_2(x) \| + \| \frac{1}{p}k_2(y) \|)$
$=\frac{1}{3}(\| x+\frac{1}{q}k_2(x)-x \| + \| y+\frac{1}{q}k_2(y)-y \|)$
$=\frac{1}{3}(\| f(e)x+\frac{1}{q}f(e)k_2(x)-x \| + \| f(e)y+\frac{1}{q}f(e)k_2(y)-y \|)$
$=\frac{1}{3}(\| f(x)-x \| + \| f(y)-y \|)$

Thus, f satisfies the Kannan condition. So, by [10], f has a unique fixed point in $\mathcal{A}$.

Theorem 3.11: Let $f:\mathcal{A}\to\mathcal{A}$ be an extended (k_1,k_2)-centralizer satisfying the condition:
$\| f(x)-f(y) \| \leq \max(\| \frac{1}{p}k_2(x) \|, \| \frac{1}{p}k_2(y) \|)$ for all $x,y\in\mathcal{A}$
and for some $p=3q, q\in\mathbb{Z}^+$. Then f has a unique fixed point if $f(e)=1$ and $k_1(e)=\frac{1}{q}$.

Proof: Similar as in Theorem 4.8.(Bianchini [4]).

The following result is based on Reich type contraction condition.

Theorem 3.12: Let f be an extended (k_1,k_2)-centralizer on a unital C^*-algebra $\mathcal{A}$. If f satisfies the conditions:

$$\| f(x)-f(y) \| \leq \frac{1}{q} \| (x-\frac{1}{\lambda}k_2(x))-(y-\frac{1}{\lambda}k_2(y)) \|,\ q>3, \lambda>0$$

then f has a unique fixed point if $f(e)=1, k_1(e)=\frac{1}{\lambda}$.

Proof. For any elements x and y in the C^*-algebra ,

$\| f(x)-f(y) \| \leq \frac{1}{q} \| (x-\frac{1}{\lambda}k_2(x))-(y-\frac{1}{\lambda}k_2(y)) \|$
$=\frac{1}{q} \| (x-x-\frac{1}{\lambda}k_2(x)+x)-(y-y-\frac{1}{\lambda}k_2(y)+y) \|$
$=\frac{1}{q} \| (x-f(e)x-k_1(e)f(e)k_2(x)+x)-(y-f(e)y-k_1(e)f(e)k_2(y)+y) \|$
$=\frac{1}{q} \| (x-f(x)+x)-(y-f(y)+y) \|$
$=\frac{1}{q} \| (x-f(x))-(y-f(y))+(x-y) \|$
$\leq \frac{1}{q} \| x-f(x) \| + \frac{1}{q} \| y-f(y) \| + \frac{1}{q} \| x-y \|$.

Also, $\frac{1}{q}+\frac{1}{q}+\frac{1}{q}<1$. So, it satisfies Reich type contraction condition and hence f has a unique fixed point.

Conclusion: *Considering different conditions for the mapping $\boldsymbol{k_2}$ and using some well known results of fixed point theory, various fixed point theorems for extended $(\boldsymbol{k_1},\boldsymbol{k_2})$-centralizer $\boldsymbol{f}$ on $\boldsymbol{C}^*$-algebras have been established. The applications of the results obtained in different aspects are a scope for further research. Moreover, extension of the results to the tensor product of $\boldsymbol{C}^*$-algebras can be investigated in future.*

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