Neutrino Oscillations: The Quantum Evidence for Massive Neutrinos

Nikasha Parashar
Sanskriti School, New Delhi
parasharnikasha@gmail.com

Abstract

Though discovered in 1930 by Wolfgang Pauli to explain a discrepancy in beta decay and first experimentally observed in 1956, neutrinos have largely remained a mystery in the years since. Neutrinos are similar to electrons, though without charge, and interact very weakly with matter. They were, for long, assumed to be massless. However, the study of the neutrino flavour oscillation - which prompted the Nobel prize in 2015 - has provided evidence for the existence of their very small masses.

Neutrino flavour oscillation is a phenomenon in which neutrinos change from one flavour(electron, tau, muon) to another as they travel through space due to the quantum mechanical mixing of their mass states. This phenomenon was first experimentally confirmed through solar neutrinos. Electron neutrinos are born as a result of nuclear fusion in the sun. However, when observed from Earth, the number of neutrinos detected was less than expected. This was the solar neutrino problem and was explained by the fact that about 2/3rd of the electron neutrinos from the sun changed their flavour as they traveled to Earth and were therefore not detected. Neutrino flavour oscillations were also observed in atmospheric neutrinos which are produced when cosmic ray particles collide with the nuclei in the earth's atmosphere, producing pions and kaons which decay into electron and muon neutrinos. The atmospheric neutrino problem refers to the discrepancy between the expected rate of electron and muon neutrino interactions and the number actually detected by detectors under the surface of the earth, namely a lower-than-expected muon neutrino interaction rate. This was explained by the fact that muon neutrinos oscillated to tau neutrinos and were therefore not detected readily. These problems are further solved using the matter effect, which describes how neutrino oscillations are significantly modified when neutrinos travel through matter, as compared to a vacuum. This is due to the interaction of neutrinos with electrons, protons and neutrons in the medium which alters their effective masses and mixing angles, thereby altering the probability of flavour oscillations.

This paper aims to derive a heuristic model for neutrino flavour oscillation by treating them as simple pendulum mechanical states, and then numerically plotting the probability of neutrino flavour oscillations as a function of length and energy. It will additionally explore solar and atmospheric neutrinos and the matter effect. The findings from this project offer a deeper understanding of neutrino oscillations as well as a model that can aid in the interpretation of experimental data and explore the fundamental nature of neutrinos and their role in the universe.

Index Terms: Neutrino flavour oscillation, solar neutrino problem, atmospheric neutrino problem, matter effect

1. Method

I will begin by deriving a master equation for the probability of neutrino flavour oscillations in vacuum and then use that to find the expression for probability of both two flavour and three flavour neutrino oscillations in vacuum. The derived equations will then be used to explain the atmospheric neutrino problem and solar neutrino problem. Finally, I will account for the matter effect and alter the expression of probability to give the expression for probability of oscillation in the presence of matter.

2. Derivation of the Master Equation

A neutrino is a state that is produced in a weak interaction. That is, it is a flavour eigenstate and is always produced with or absorbed to give a charged lepton or muon, tau or electron flavour. The neutrino generated with a charged electron is an electron neutrino and so on. However, the flavour eigenstates are not identical to the mass eigenstate, and are in fact coherent superpositions of the mass eigenstates. So, if we generate a neutrino, it will have definite flavour ν_α but will be produced as a linear combination of the states of definite masses. That is:

$$|\nu_\alpha >= \sum_{i=1}^{3} U_{\alpha k}|\nu_k >$$

Or

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad (1)$$

Where U= $\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$. U is a mixing matrix knows as the Pontecorvo-Maka-Nakagawa-Sakata (PMNS) which is complex, unitary and describes how flavour eigenstates are related to mass eigenstates. Unitary means that

$$U^\dagger U = I$$
$$U^\dagger = U^{-1} = (U^*)^T$$

From which we get

$$\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1}^* & U_{\mu1}^* & U_{\tau1}^*\\ U_{e2}^* & U_{\mu2}^* & U_{\tau2}^*\\ U_{e3}^* & U_{\mu3}^* & U_{\tau3}^*\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \quad (2)$$

The unitary of U gives us that

$$\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}\begin{pmatrix}U_{e1}^* & U_{\mu1}^* & U_{\tau1}^*\\ U_{e2}^* & U_{\mu2}^* & U_{\tau2}^*\\ U_{e3}^* & U_{\mu3}^* & U_{\tau3}^*\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

So,

$$U_{e1}U_{e1}^* + U_{\mu1}U_{\mu1}^* + U_{\tau1}U_{\tau1}^* = 1$$
$$U_{e2}U_{e2}^* + U_{\mu2}U_{\mu2}^* + U_{\tau2}U_{\tau2}^* = 1$$
$$U_{e3}U_{e3}^* + U_{\mu3}U_{\mu3}^* + U_{\tau3}U_{\tau3}^* = 1$$
$$U_{e1}U_{\mu1}^* + U_{e2}U_{\mu2}^* + U_{e3}U_{\mu3}^* = 0$$
$$U_{e1}U_{\tau1}^* + U_{e2}U_{\tau2}^* + U_{e3}U_{\tau3}^* = 0$$
$$U_{\mu1}U_{\tau1}^* + U_{\mu2}U_{\tau2}^* + U_{\mu3}U_{\tau3}^* = 0$$

If at time t=0, we create a neutrino in pure state |□□ > state

$$|\psi(x=0)>= U_{\alpha1}|\nu_1> + U_{\alpha2}|\nu_2> + U_{\alpha3}|\nu_3> \quad (3)$$

After travelling distance L, (assuming the neutrinos are relativistic that is they move with speeds close to the speed of light and their kinetic energy is much greater than their rest mass energies) the wave function evolves to become

$$|\psi(x=0)>= U_{\alpha1}|\nu_1> e^{-i\phi_1} + U_{\alpha2}|\nu_2> e^{-i\phi_2} + U_{\alpha3}|\nu_3> e^{-i\phi_3} \quad (4)$$

Where ϕ = Et-pL = (E-p)L if the neutrino is relativistic

$$E^2 = p^2 + m^2$$
$$= p^2(1+\frac{m^2}{p^2})$$
$$E = p\,(1+\frac{m^2}{p^2})^{1/2}$$
$$E = p + \frac{m^2}{2p^2} \quad (5)$$

So,

$$\phi_i = (E_i - p_i)L = \frac{{m_i}^2}{2{p_1}^2}L = \frac{{m_i}^2}{2E_i}L \quad (6)$$

Now, the probability that a neutrino initially of flavour ν_α is detected as a neutrino of flavour by a detector at a distance L is

$$P(\nu_\alpha \to \nu_\beta) = |< \nu_\beta|\psi(L) > |^2$$
$$=(U_{\alpha 1}e^{-i\phi_1} \langle \nu_\beta|\nu_1 \rangle + U_{\alpha 2}e^{-i\phi_2}\langle \nu_\beta|\nu_2\rangle + U_{\alpha 3}e^{-i\phi_3}\langle \nu_\beta|\nu_3\rangle)^2$$

Using the orthonormality of the mass eigenstates (< □□|□□ >= □*□□)

$$P(\nu_\alpha \to \nu_\beta) = \left(U_{\alpha 1}U^*_{\beta 1}e^{-i\phi_1} + U_{\alpha 2}U^*_{\beta 2}e^{-i\phi_2} + U_{\alpha 3}U^*_{\beta 3}e^{-i\phi_3}\right)^2 \quad (7)$$

Using the fact that

$$|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2R(z_1 z_2^* + z_1 z_3^* + z_2 z_3^*)$$

We get

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right)\sin^2\left(\Delta m^2_{ij}\frac{L}{4E}\right)$$
$$+2\sum_{i>j} \mathrm{Im}\left(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right) sin\left(\Delta m^2_{ij}\frac{L}{2E}\right) \quad (8)$$

Where $\delta_{\alpha\beta}$ is known as the Kronecker delta and accounts for the case where the flavour of the neutrino before and after oscillation is the same. It is equal to 1 when α=β and 0 when α≠β

We have derived the master formula - Equation (8) - which can now be used to formulate the probability for 2 and 3 flavour neutrino oscillations

3. Two Flavour Oscillations in Vacuum

3.1. Derivation of the oscillation probability

As found in section 2,

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right)\sin^2\left(\Delta m^2_{ij}\frac{L}{4E}\right)$$
$$+2\sum_{i>j} \mathrm{Im}\left(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\right) sin\left(\Delta m^2_{ij}\frac{L}{2E}\right) \quad (8)$$

For 2 flavour oscillations, the mixing matrix U is a 2x2 matrix defined as

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Setting i=2 and j=1, the terms of importance from (8) can be re written as

$$\text{The Real Part} = \mathrm{Re}\left(U^*_{\alpha 2}U_{\beta 2}U_{\alpha 1}U^*_{\beta 1}\right)sin^2$$

$$= \mathrm{Re}(-sin^2\theta cos^2\theta)$$

The Imaginary Part = Im(□$^*_{□2}$□$_{□2}$□$_{□1}$□$^*_{□1}$)

= 0 (assuming there is no CP violation)

Equation (8) can now be rewritten as

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} + 4sin^2\theta cos^2\theta sin^2\left(\Delta m^2 \frac{L}{4E}\right)$$

Where $\Delta m^2$is the mass square difference between the two eigenstates

For α≠β, $\delta_{\alpha\beta} = 0$ and $sin^2(2\theta) = 4sin^2\theta cos^2\theta$ so

$$P(\nu_\alpha \to \nu_\beta) = sin^2(2\theta)\sin^2\left(\Delta m^2 \frac{L}{4E}\right) \quad (9)$$

If we measure L in km and E in GeV and address the $\hbar$ and c left out,

$$P(\nu_\alpha \to \nu_\beta) = sin^2(2\theta)\, sin^2(1.27\Delta m^2 \frac{L(km)}{E(GeV)}) \quad (10)$$

This is the probability that a neutrino initially of flavour ν_α is detected as a neutrino of flavour by a detector at a distance L. The probability of generating ν_α and also detecting ν_α is (1- $P(\nu_\alpha \to \nu_\beta)$) in the 2 flavour case

3.2. Some Important Observations

1. For oscillation to occur, $P(\nu_\alpha \to \nu_\beta) \neq 0$, which means $\Delta m^2 \neq 0$ that is, at least one of the mass states must be non-zero. That means, for oscillations to happen, neutrinos must have mass. This simple statement was a turning point in the understanding of neutrinos, since up until this proof, neutrinos were assumed to be massless.
2. The mixing angle θ: It describes how different flavour states are from mass states. If □ $= 0$ then flavour states and mass states are identical, that is no oscillations will take place. If □ $= \frac{□}{4}$, the mixing is maximum, and at some point in time, all neutrinos of flavour ν_α will have oscillated to ν_β
3. L and E are parameters controlled by experimentalists, where L is the distance from the source to the detector and E is the energy of the neutrino generated. For the maximum sensitivity to the oscillation,

$$sin^2(1.27\Delta m^2 \frac{L(km)}{E(GeV)})=1$$

That is,

$$1.27\Delta m^2 \frac{L(km)}{E(GeV)} = \frac{\pi}{2}$$

$$\frac{L(km)}{E(GeV)} = \frac{\pi}{2.54\Delta m^2}$$

4. Solar Neutrinos

Electron neutrinos are born as a result of nuclear fusion in the sun. However, when observed from Earth, the number of neutrinos detected was less than expected. This was termed as the solar neutrino problem and was what first prompted the idea of neutrino flavour oscillation. A number of solar neutrino experiments were conducted in the mid to late 1900s such as Homestake, Kamiokande and Super-Kamiokande using a variety of detection methods. However, all of them were only able to identify that there were fewer electron neutrinos than expected but none offered proof of explanation on why this happened. This was until the Sudbury Neutrino Observatory (SNO) experiment in 1999. Unlike other experiments, SNO used a detection method involving heavy water (D_2O) target which contains deuterium, a heavier isotope of hydrogen. Deuterium has a neutron in addition to a proton, allowing for different types of interactions with neutrinos. It was designed to detect neutrinos through multiple channels such as

- Charged current interactions - These when neutrinos interact with a neutron in the heavy water, converting it into a proton and an electron. This process is specific to electron neutrinos (ν_e).
- Neutral current Interactions: These involve a neutrino interacting with a neutron, which turns it into a proton and a neutrino (of any flavour), without changing the charge. This allowed SNO to detect neutrinos of any flavour ($\nu_{\square}$, ν_μ, or ν_τ) unlike previous experiments.

By measuring all flavour of neutrinos through neutral and charged current interactions, and comparing it with the number of electrons coming from the sun, SNO was able to confirm the phenomenon of neutrino oscillations.

Later, when proof of neutrino flavour oscillation was established and the known parameters L = 10^8km, $\Delta m^2 \sim 10^{-10}\ eV^2$ and E ~ 10 MeV were input into the probability expression, it was confirmed that 2/3 of electron neutrinos from the sun were oscillating to muon and tau neutrinos on their way to the earth.

5. Atmospheric Neutrinos

5.1. Introduction

Cosmic rays are a radiation of high energy particles entering earth's atmospheric from the surrounding universe. These are composed of 95% photons, 5% alpha particles and <1 % electrons and heavier nuclei. These particles enter earth's atmosphere and interact with the nuclei there. In these interactions, many π mesons, and less abundantly K mesons are produced. These unstable mesons decay into other particles which are often also unstable and further decay. For example, a π^+ decays to a ν_μ and a muon (μ^+) which further decays to a positron (e^+), a □̄$_\mu$ and a ν_e as shown below. A similar interaction takes place for π^- and K mesons.

$$\pi^+ \rightarrow \mu^+ \nu_\mu \qquad \mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$$

$$\pi^+ \rightarrow \mu^- \nu_\mu \qquad \mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$$

5.2. Atmospheric Neutrino Problem

From the decay equations given above, it can be seen that the ratio

$$R = \frac{\nu_\mu + \bar{\nu}_\mu}{\nu_e + \bar{\nu}_e}$$

should be 2. As a matter of fact, computers that model the entire process predict R=2 with an uncertainty of 5%. In atmospheric neutrino experiments, a detector is positioned on or just below the earth's surface and the ratio R is slightly modified to become

$$R = \frac{(N\mu/Ne)_{DATA}}{(N\mu/Ne)_{SIM}}$$

Where □μ is the total number of ν_μ interacting with the detector and □□ is the total number of $\nu_□$ events interacting with the detector. The ratio of these 2 is calculated both in data and in computation models. If the observed and expected flavour compositions align then R=1. However, in atmospheric neutrino experiments such as Kamiokande, Super-Kamiokande the value of R is significantly less than 1, meaning that there were less ν_μ in the data than the prediction or there were more $\nu_□$, or a combination of both. This came to be known as the atmospheric neutrino problem.

5.3. Explanation of the Atmospheric Neutrino Problem

Later experiments found that the discrepancy in the value of R was due to the fact that there were lower than expected ν_μ in the data. The fact that there was a deficiency in ν_μ but no increase in $\nu_{\square}$ meant that the primary oscillation was $\nu_\mu \rightarrow \nu_\tau$.

For detectors placed on or just below the surface of earth, as they are in all atmospheric neutrino experiments, neutrinos can reach them from all sides. So, as observed in the Figure 1 below, neutrinos coming from up only travel around 10-15km while those coming from down travel up to 13000km.

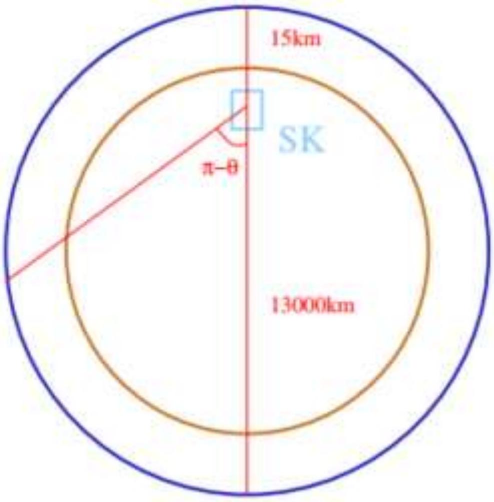

Figure 1

If we now use these parameters to calculate the probability of oscillation in each case:

1. For neutrinos coming from straight up

   L ~ 15km

   $\Delta m^2 \sim 10^{-3}\ eV^2$ (a reasonable assumption)

   E ~ 10 GeV

   If we input these value in (9)

$$P(\nu_\mu \rightarrow \nu_\tau) \sim 0$$

   Meaning no neutrinos oscillate in this case

2. For neutrinos coming from straight down

   L ~ 13000km

   $\Delta m^2 \sim 10^{-3}\ eV^2$

$$P(\nu_\mu \rightarrow \nu_\tau) \sim 0.5$$

   Meaning half the muon neutrinos coming from down were oscillating to tau neutrinos

The fact that muon neutrinos were oscillating to tau neutrinos is what finally resolved the atmospheric neutrino problem

6. Matter Effect

6.1. Introduction

Neutrinos oscillate due to a difference in phase between the wave packets of their mass eigenstates. In vacuum, this phase difference occur due to wave packets that propagate with different velocities due to mass differences. In matter, however, the phase difference determined the total energy of the state, which is E+V if the neutrinos is propagating in a potential of V. If this potential, V, is different for different neutrino flavours, then a phase difference is introduced and the neutrino now oscillate through matter effects.

6.2. Derivation of probability of oscillation in matter

As established in Equation (5), for neutrino in vacuum with momentum p

$$E = p + \frac{m^2}{2p^2}$$

If a neutrino beam with neutrino of definite flavour α at a space time point (x, t) = (0, 0) is aimed along the x axis and the neutrinos are allowed to propagate through free space towards a detector at distance L then the □$_{1,2}$ propagate according to the time independent Schrodinger equation is

$$i\frac{\partial}{\partial t}|\nu_i(x,t) >= E|\nu_i(x,t) >= -\frac{1}{2m_i}\frac{\partial^2}{\partial x^2}|\nu_i(x,t) > i\exists 1{,}2 \qquad (11)$$

The solution to which is the plane wave

$$\left|\nu_i(x,t) >= e^{-i(E_i t - p_i x)}\right|\nu_i(0{,}0) > \qquad (12)$$

Equation (12) shows the time dependence of mass eigenstates. Differentiating with respect to time and ignoring common phase factor □$^{□(□_□ □)}$, we obtain

$$i\frac{d}{dt}\binom{\nu_1}{\nu_2} = H\binom{\nu_1}{\nu_2} \qquad (13)$$

Where H is known as the Hamiltonian operator and in vacuum is

$$H = \frac{1}{2E}\begin{pmatrix} {m_1}^2 & 0 \\ 0 & {m_2}^2 \end{pmatrix} \qquad (14)$$

Taking the usual 2x2 matrix for U

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We can write flavour states in terms of their mass states as

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

Using the unitary of U

$$U^\dagger \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

$$i\frac{d}{dt} U^\dagger \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = HU^\dagger \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}$$

$$i\frac{d}{dt} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = UHU^\dagger \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix}$$

Where, the transformed Hamiltonian becomes

$$H_f = UHU^\dagger = \frac{{m_1}^2 + {m_2}^2}{2E} 1 + \frac{\triangle m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \tag{15}$$

Where 1 is a 2x2 unit matrix. Let $\square_0 = \frac{{\square_1}^2 + {\square_2}^2}{2\square} 1$. To account for potential, a potential term now has to be added to the equation to become

$$i\frac{d}{dt} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = (H_f + V) \begin{pmatrix} \nu_\alpha \\ V_\alpha \end{pmatrix} \tag{16}$$

Taking V_α as the potential experienced by ν_α and V_β and the potential experienced by ν_β

$$i\frac{d}{dt} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = (H_0 + V) \begin{pmatrix} -\frac{\triangle m^2}{4E}\cos 2\theta + V_\alpha & \frac{\triangle m^2}{4E}\sin 2\theta \\ \frac{\triangle m^2}{4E}\sin 2\theta & \frac{\triangle m^2}{4E}\cos 2\theta + V_\beta \end{pmatrix} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} \tag{17}$$

This equation can be slightly modified to become

$$i\frac{d}{dt} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = (H_0 + V) \begin{pmatrix} -\frac{\triangle m^2}{4E}\cos 2\theta + (V_\alpha - V_\beta) & \frac{\triangle m^2}{4E}\sin 2\theta \\ \frac{\triangle m^2}{4E}\sin 2\theta & \frac{\triangle m^2}{4E}\cos 2\theta \end{pmatrix} \begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} \tag{18}$$

Using

$$i\frac{d}{dt}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = U^\dagger i\frac{d}{dt}\begin{pmatrix}\nu_\alpha\\ \nu_\beta\end{pmatrix}$$

We get

$$i\frac{d}{dt}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = U^\dagger\left[H_0 + \begin{pmatrix}-\frac{\Delta m^2}{4E}\cos 2\theta + (V_\alpha - V_\beta) & \frac{\Delta m^2}{4E}\sin 2\theta\\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta\end{pmatrix}\right]\begin{pmatrix}\nu_\alpha\\ \nu_\beta\end{pmatrix}$$
$$= \left(U^\dagger\left[H_0 + \begin{pmatrix}-\frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta\\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta\end{pmatrix}\right]U + U^\dagger\begin{pmatrix}V_\alpha - V_\beta & 0\\ 0 & 0\end{pmatrix}U\right]\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$$
$$= \frac{1}{2E}\left[\begin{pmatrix}m_1^2 & 0\\ 0 & m_2^2\end{pmatrix} + \begin{pmatrix}\Delta V\cos\cos^2\theta & \Delta V\cos\theta\sin\theta\\ \Delta V\cos\theta\sin\theta & \Delta V\sin^2\theta\end{pmatrix}\right]\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$$
$$= \left[\frac{1}{2E}\begin{pmatrix}m_1^2 + \Delta V\cos^2\theta & \Delta\,\mathrm{V}\cos\theta\sin\theta\\ \Delta\,\mathrm{V}\cos\theta\sin\theta & m_2^2 + \Delta\,\mathrm{V}\sin\theta\end{pmatrix}\right]\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \quad (19)$$

Where $\Delta V = V_\alpha - V_\beta$

Since this matrix is not diagonalised, we need to diagonalise it in order to get the correct eigenstates. When diagonalised, if m_{1m} and m_{2m} are the modified mass eigenvalues in matter then

$$m_{1m,2m}^2 = \frac{1}{2}\left[(m_1^2 + m_2^2 + \Delta V) \pm \sqrt{(\Delta V - \Delta m^2\, cos\, cos\, 2\theta\,)^2 + (\Delta m^2)^2 2\theta}\right] \quad (20)$$

and

$$\Delta m_m{}^2 = m_{1m}^2 - m_{,2m}^2 = \Delta m^2\sqrt{(\Delta V/\, m^2 - cos\, 2\theta)^2 + sin^2\, 2\theta} \quad (21)$$

We perform a standard oscillation analysis to find that

$$sin2\theta_m = \frac{sin2\theta}{\sqrt{(\Delta V/\, m^2 - cos\, 2\theta)^2 + sin^2\, 2\theta}} \quad (22)$$

We can now express the probability of oscillation in matter as

$$\mathrm{P}(\nu_\alpha \rightarrow \nu_\beta) = sin^2(2\theta_m)\, sin^2(1.27\, \Delta m_m{}^2\, \frac{L(km)}{E(GeV)}\,) \quad (23)$$

Which is the same expression as probability of oscillation in vacuum, except in terms of the mass eigenstates in matter and a modified mixing angle.

6.3. Some Important Observation

1. If $\Delta V = 0$ (that is, there is no matter to provide a potential difference) then ${\Delta m_m}^2 = \Delta m^2$ and $sin2\theta_m = sin2\theta$, meaning in absence of matter, the matter modified parameters reduce to vacuum parameters.
2. If $sin2\theta = 0$ then $sin2\theta_m = 0$ meaning there must be oscillation in vacuum for there to be oscillations in matter.
3. If the matter is very dense then $\Delta V \to \infty$ and $sin2\theta_m \to 0$ meaning there is no oscillation in very dense matter
4. MSW Resonance: if $\frac{\Delta V}{m^2} = cos2\theta$ then the $sin2\theta_m$, the matter mixing angle is 1 meaning even if vacuum mixing is tiny, there is chance that 100% of neutrinos oscillate.

7. Three Flavour Neutrino Oscillations in Vacuum

In the 3 flavour case, the mixing matrix U is 3x3 matrix of form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{i\delta_{CP}} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which when multiplied out becomes

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix}$$

Where c and s stand for cos and sin respectively. From (8), we already know that

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right)$$
$$+2\sum_{i>j} \mathrm{Im}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) sin\left(\Delta m_{ij}^2 \frac{L}{2E}\right)$$

In this case, if we assume $\delta_{\square\square} = 0$ then imaginary term vanishes, leaving us with

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}\left(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\right) \sin^2\left(\Delta m_{ij}^2 \frac{L}{4E}\right) \quad (24)$$

However, in the three flavour case, we have 3 neutrino mass eigenstates to take into account with the mass splitting's being defined by

$$m_{12}^2 + m_{23}^2 + m_{31}^2 = 0$$

Where it has been experimentally observed that $m_{12}^2 = 8x10^{-5}eV^2$ and $m_{23}^2 = 3x10^{-3}eV^2$. Since $m_{31}^2 = -m_{13}^2$ and m_{12}^2 is so small, it can be reasonably assumed that $m_{13}^2 \approx m_{23}^2$. We can now write the probability expression as

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= -4\sum_{i>j} (U_{\alpha i}U_{\beta i}U_{\alpha j}U_{\beta j})\sin^2\left(1.27\Delta m_{ij}^2 \frac{L}{E}\right) \\ &= -4\left[(U_{\alpha 1}U_{\beta 1}U_{\alpha 2}U_{\beta 2})\sin^2\left(1.27\Delta m_{12}^2 \frac{L}{E}\right)\right. \\ &+(U_{\alpha 1}U_{\beta 1}U_{\alpha 3}U_{\beta 3})\sin^2\left(1.27\Delta m_{13}^2 \frac{L}{E}\right) \\ &\left.+(U_{\alpha 2}U_{\beta 2}U_{\alpha 3}U_{\beta 3})\sin^2\left(1.27\Delta m_{23}^2 \frac{L}{E}\right)\right] \end{aligned} \quad (25)$$

We now consider 2 cases:

1. When L/E is small –

If L/E is small, then $1.27\Delta m_{12}^2 \frac{L}{E}$ is also very small and

$$\left(1.27\Delta m_{12}^2 \frac{L}{E}\right) \to 0$$

Keeping with the assumption that $m_{13}^2 \approx m_{23}^2$, we can re write the probability expression to get

$$P(\nu_\alpha \to \nu_\beta) = -4[U_{\alpha 1}U_{\beta 1}U_{\alpha 3}U_{\beta 3} + U_{\alpha 2}U_{\beta 2}U_{\alpha 3}U_{\beta 3}]\left(1.27\Delta m_{23}^2 \frac{L}{E}\right)$$

Using the mixing matrix U, we find that

$$P(\nu_\mu \to \nu_\tau) = \cos^4(\theta_{13})\sin^2(2\theta_{23})\sin^2\left(1.27\Delta m_{23}^2 \frac{L}{E}\right)$$
$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta_{13})\sin^2(\theta_{23})\sin^2\left(1.27\Delta m_{23}^2 \frac{L}{E}\right)$$
$$P(\nu_e \to \nu_\tau) = \sin^2(2\theta_{13})\cos^2(\theta_{23})\sin^2\left(1.27\Delta m_{23}^2 \frac{L}{E}\right)$$

2. When L/E is large –

The terms with Δm_{23}^2 and m_{13}^2 oscillate rapidly to average out to 0.5 so the probability expression becomes

$$P(\nu_\alpha \to \nu_\beta) = -4[(U_{\alpha1}U_{\beta1}U_{\alpha2}U_{\beta2})sin^2\left(1.27\Delta m_{12}^2\frac{L}{E}\right) + \frac{1}{2}(U_{\alpha1}U_{\beta1}U_{\alpha3}U_{\beta3} + U_{\alpha2}U_{\beta2}U_{\alpha3}U_{\beta3})]$$

8. Conclusion

The study of neutrinos continues to be one of the most fascinating fields in physics, mostly because it constantly pushed the boundaries of the Standard Model. There have been dozens of experiments designed especially to learn more about these tiny particles, such as the Homestake, Kamiokande and Super-Kamiokande already mentioned as well as the Daya Bay, RENO and Double Chooz experiments which aimed to measure θ_{13} to help understand neutrino mass hierarchies.

Yet, despite knowing of the existence of neutrinos for a long time, there has been shockingly little ground covered when it comes to unravelling the mysteries around them. While we do know two of the three mass-squared differences ($\triangle m_{12}^2$ and $\triangle m_{23}^2$), the third mass square difference, the sign of $\triangle m_{23}^2$ as well the value of the absolute masses remains a mystery. Additionally, even ambiguity in the neutrino mass hierarchy—whether the mass of the third neutrino is larger or smaller than the others—continues to be a critical puzzle, as does the value of δ_{cp}.

The future in this field is aimed towards answering these questions, and the possibilities are endless. New experiments promise new answers that, for all we know, might change the face of physics as we know it.

9. References

1. Unknown, U. (2020). *Neutrino oscillations*. https://warwick.ac.uk/fac/sci/physics/staff/academic/boyd/stuff/neutrinolectures/lec_oscillations.pdf
2. Class for Physics of the Royal Swedish Academy of Sciences. (2015). *Scientific Background on the Nobel Prize in Physics 2015*. https://www.nobelprize.org/uploads/2017/09/advanced-physicsprize2015.pdf
3. *Solar neutrinos | All Things Neutrino*. (n.d.). https://neutrinos.fnal.gov/sources/solar-neutrinos/
4. *Solving the mystery of the missing neutrinos*. (n.d.). NobelPrize.org. https://www.nobelprize.org/prizes/themes/solving-the-mystery-of-the-missing-neutrinos/
5. LoSecco, J. M. & Physics Department, University of Notre Dame, Notre Dame, IN 46556, USA. (n.d.). Problems with Atmospheric Neutrino Oscillations.. https://lss.fnal.gov/conf2/C990817/h4_2_23.pdf
6. *997foap.conf..467L page 467*. (n.d.). https://adsabs.harvard.edu/full/1997foap.conf..467L
7. Kajita, T. (2012). Atmospheric neutrinos. *Advances in High Energy Physics*, *2012*, 1–24. https://doi.org/10.1155/2012/504715

8. Choubey, S. (2016). Atmospheric neutrinos: Status and prospects. *Nuclear Physics. B*, *908*, 235–249. https://doi.org/10.1016/j.nuclphysb.2016.03.026
9. Bibek, K., & Husain, D. S. (2022). Oscillation of neutrino in a vacuum with mixing flavor. *International Journal of Physics Research and Applications*, *5*(1), 018–026. https://doi.org/10.29328/journal.ijpra.1001046
10. Kajita, T. (2012). Atmospheric neutrinos. *Advances in High Energy Physics*, *2012*, 1–24. https://doi.org/10.1155/2012/504715
11. Wan, L. (2023, December 20). *Atmospheric neutrinos revisited*. Physics. https://physics.aps.org/articles/v16/212
12. Kajita, T. (2010). Atmospheric neutrinos and discovery of neutrino oscillations. *Proceedings of the Japan Academy Series B*, *86*(4), 303–321. https://doi.org/10.2183/pjab.86.303