Conversion of the Volterra Integral Equation of the First Kind to the Volterra Integral of the Second Kind

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Abstract: Integral Equations are considered significantly as important topics regarding to (Pure and Applied Mathematics). This importance comes to the prominence because these equations are taken a critical and a vital role of physical and chemical problems.In this paper we present Conversion of Volterra Integral Equation of the first Kind in to the Volterra Integral Equation of the second kind .We solve a few examples of this type of conversion, and the results generate convergent solutions for the resolved kernel.

Keywords: Integral equations, Volterra Integral Equation (VIE) of the first Kind, VIE of the second kind, Ordinary Differential Equations(ODEs)

Introduction:
One of the most important mathematical topics in applied as well as pure mathematics is the study of integral equations.[1-6]. It has a wide range of occupies in many physical problems. It is easy to convert many of the initially and boundary-value problems related to ordinary differential equations (ODEs) and partial differential equations (PDEs) into problems relating to the approximation solutions of particular integral equations. Fourier integrals were studied in order to introduce integral equations to the world [13–17].Abel created a second integral equation in 1826. The works of Italian mathematician V. Volterra (1896) represented the beginning of the actual development of the theory of integral equations [23, 24]. Applications in stereology, conduction of heat, electromagnetic radiation from a semi-infinite solid, mathematical physics, and crystal growth give conception to VIEs [18, 19, 20].

Consider the VIE

$$\boldsymbol{\alpha(x)u(x) = P(x) + \mu \int_a^x M(x,\beta)u(\beta)d\beta} \quad \ldots(1)$$

The function $\boldsymbol{M(x,\beta)}$ is the kernel of Volterra equation . When $\propto = \mathbf{0}$ the equation

$$\mathbf{P(x) + \mu \int_a^x M(x,\beta)u(\beta)d\beta = 0} \quad \ldots \quad (2)$$

Is known as the VIE of first kind..When $\propto = \mathbf{1}$ the integralequation

$$\boldsymbol{u(x) = P(x) + \mu \int_a^x M(x,t)u(\beta)d\beta} \quad \ldots(3)$$

such equation is called VIE of the Second Kind

Leibnitz rule for differentiation under integral sign:

Let the given function $\boldsymbol{F(z) = \int_{a(z)}^{b(z)} f(x,z)dx}$ where$\boldsymbol{a(z), b(z)}$are the function of z or constant then

$$\boldsymbol{\frac{dF(z)}{dz} = \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z} dx + f(b(z),z)\frac{db(z)}{dz} - f(a(z),z)\frac{da(z)}{dz}} \quad (4)$$

Method for conversion :

Consider the VIE of the first Kind when $\boldsymbol{\mu = -1}$

$$\boldsymbol{P(x) = \int_a^x M(x,\beta)y(\beta)d\,\beta} \quad (5)$$

where$\boldsymbol{y(\beta)}$ is known function Suppose the kernel $\boldsymbol{M(x,\beta)}$ and$\frac{\partial M(x,\beta)}{\partial x}$ be a continuous for $\boldsymbol{a \le x \le b}$, $\boldsymbol{a \le \beta \le b}$$\boldsymbol{P(a) = 0}$is the necessary condition for the equation (5)

D.w.r.t.x equation (5)

$$\int_a^x \boldsymbol{M'(x,\beta)y(\beta)d\beta + M(x,x)y(x)\frac{d}{dx}(x) - M(x,a)y(a)\frac{d}{dx}(a) = P'(x)}$$

$\int_a^x \boldsymbol{M'(x,\beta)y(\beta)d\beta + M(x,x)y(x) = P'(x)}$ (6)

Let $\boldsymbol{M(x,x) \neq 0}$ at any point in $\boldsymbol{[a,b]}$, using equation (6)

$$\boldsymbol{y(x) = \frac{P'(x)}{M(x,x)} - \int_a^x \frac{M'(x,\beta)}{M(x,x)} y(\beta)d\beta}$$

it is the VIE of the Second Kind

Let $\boldsymbol{M(x,x) = 0}$ at any point in $\boldsymbol{[a,b]}$ then

$$\int_a^x \boldsymbol{M'(x,\beta)y(\beta)d\beta = P'(x)}$$

Assume that $\frac{\partial^2 M(x,\beta)}{\partial x^2}$ is continuous in $\boldsymbol{a \le x \le b}$

$$\int_a^x \boldsymbol{M''(x,\beta)y(\beta)d\beta + M'(x,x)y(x)\frac{d}{dx}(x) - M'(x,a)y(a)\frac{d}{dx}(a) = P''(x)}$$

$$\int_a^x \boldsymbol{M''(x,\beta,)y(\beta)d\beta + M'(x,x)y(\beta) = P''(x)}$$

$\boldsymbol{If\ M'(x,x) \neq 0}$for any$\boldsymbol{x}$ in $\boldsymbol{[a,b]}$ then

$$\boldsymbol{y(x) = \frac{P''(x)}{M'(x,x)} - \int_a^x \frac{M''(x,\beta)}{M'(x,x)} y(\beta)d\,\beta}$$

it is the VIE of the Second Kind

Continue this process we get if $\boldsymbol{M^n(x,\beta)}$ is continuous then $\boldsymbol{P^n(x)}$ is also continuous

$$\int_a^x \boldsymbol{M^n(x,\beta)y(\beta)d\beta + M^{n-1}(x,x)y(\beta) = P^n(x)}$$

If $\boldsymbol{M^{n-1}(x,x) \neq 0}$then

$\boldsymbol{y(x) = \frac{P^n(x)}{M^n(x,x)} - \int_a^x \frac{M^n(x,\beta)}{M^{n-1}(x,x)} y(\beta)d\beta}$ (7)

it is the VIE of the Second Kind

Example1: $\boldsymbol{\int_0^x e^{x-\beta}y(\beta)d\beta = sinx}$,VIE of first kind ,

Here$\boldsymbol{P(x) = sinx,\ M(x,\beta) = e^{x-}}$, Dw.r.t.$\boldsymbol{x}$by using Leibnitz rule we get

$$\int_0^x \boldsymbol{e^{x-\beta}y(\beta)d\beta + e^{x-x}y(x)\frac{dx}{dx} - e^{x-0}y(0)\frac{d0}{dx} = cosx}$$

$$\int_0^x e^{x-\beta} y(\beta) d\beta + y(x) = cosx$$

$y(x) = cosx - \int_0^x e^{x-\beta} y(\beta) d\beta$, here $\mu = -1, P(x) = cosx, M(x,\beta) = e^{x-\beta}$

We have $M_1(x,\beta) = M(x,\beta) = e^{x-\beta}$

$$M_2(x,\beta) = \int_t^x e^{x-\beta} d\beta = e^{x-\beta}(x-\beta)$$

$$M_3(x,\beta) = \int_t^x M(x,\beta) M_2(x,\beta) d\beta = e^{x-\beta} \frac{(x-\beta)^2}{2!}$$

$$M_4(x,\beta) = \int_t^x M(x,\beta) M_3(x,\beta) d\beta = e^{x-\beta} \frac{(x-\beta)^3}{3!}$$

$$M_n(x,\beta) = \int_t^x M(x,\beta) M_{n-1}(x,\beta) d\beta = e^{x-\beta} \frac{(x-\beta)^{n-1}}{(n-1)!}$$

Resolvent $R(x,\beta,\mu) = \sum_{n=0}^{\infty} \mu^n M_{n+1}(x,\beta) = \sum_{n=0}^{\infty} (-1)^n e^{x-\beta} \frac{(x-\beta)^n}{n!} = 1$

Solution of equation becomes $u(x) = P(x) + \mu \int_a^b R(x,y,\mu) P(y) dy = cosx - sinx$

Example2: Consider the $\frac{23}{6}x = -\frac{1}{8}\int_0^1 xyu(y)dy$VIE of first kind ,

$$M(x,y) = xy, P(x) = \frac{23}{6}x, \mu = \frac{1}{8}$$

$$M_1(x,y) = xy,\ M_2(x,\beta) = \int_0^1 M(y',y) dy' = \frac{xy}{3}$$

$$M_3(x,y) = \int_0^1 M(x,y') M_2(y',y) dy' = \frac{xy}{3^2}$$

$$M_n(x,y) = \frac{xy}{3^{n-1}}$$

Resolvent

$$R(x,\beta,\mu) = \sum_{n=0}^{\infty} \mu^n M_{n+1}(x,t) = \frac{24}{23}xy$$

Solution of equation becomes $u(x) = P(x) + \mu \int_a^b R(x,y,\mu) P(y) dy = 4x$

Conclusion:

In this paper, we classify VIEs, specifically the first and second kinds. The first kind of VIE is transformed into the second kind of VIE. Resolvent kernel and approximative solutions are obtained when the VIE of the first kind is converted to the VIE of the second kind. In the resolvent kernel, increasing successive terms can increase the accuracy of the results.

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