

A Brief Review of Differential Geometry of Manifolds

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Abstract- Riemannian geometry is the study of manifolds endowed with Riemannian matrices which are roughly speaking, rules for measuring lengths of lengths of tangent vectors and angles between them. It is the most "geometric" branch of differentiable geometry: This paper gives an overview about the tools we use to understand how to adapt concepts such as the distance between two points, the angle between two crossing curves curvature of a plane curve, to a surface.

Keywords: Riemannian geometry, Levi-Civita connection, Differential geometry, Riemannian curvature, Parallel transport, General relativity.

1. Introduction

Shape is an interesting subject which has stimulated the imagination of many people. It suffices to look to become curious. Euclid did just that and come up with the first pure creation. In particular, mathematicians started wondering whether Euclid's "obvious" absolute postulates were indeed obvious and/or absolute. Scientists observed that shape and space are two closely related concepts asked whether they really look the way our senses tell us. We are studying about shape, space and some particular ways of studying them.

Since its starting, the differential and integral Calculus proved to a very versatile tool in dealing with previously untouchable problems. In the early days of geometry, nobody worried about the natural context in which the methods of calculus, feel at home." There was no need to address this aspect since for the particular problems studied this was a non-issue. As mathematics progressed as a whole the "natural context" mentioned above crystallized in the minds of mathematicians and it was a nation so important that it had to be given a name. The geometric objects which can be studied using the methods of calculus were called smooth manifolds Special cases of manifolds are the curves and the surfaces and these were quite well understood. B. Riemann was the first to note that the low dimensional ideas of this time were particular aspects of a higher dimensional world.

2. Preliminaries

In this section we introduce the tangent bundle TM of a differentiable manifold M . Intuitively this is the object we get by glueing at each point p belongs to M the corresponding Tangent space T_pM . The differentiable structure on M induces a differentiable structure on TM making it into a differentiable manifold of $\dim 2n$. The tangent bundle TM is the most important example of what is called a vector bundle over M .

2.1 Definition. Let E and M be topological manifold and $\pi \rightarrow M$ be a continuous ontomap. The triple (E, M, π) is called an n -dimensional topological vector bundle over M if

- (i) For each $p \in M$ the fibre $E_p = \pi^{-1}(p)$ is an- n dimensional vector space.
- (ii) For each $p \in M$ there exist a bundle chart $(\pi^{-1}(v), \phi)$ consisting of the preimage $\pi^{-1}(v)$ of an open neighbourhood v of p and a homeomorphism $\phi : \pi^{-1}(v) \rightarrow V \times \mathbb{R}^n$ such as for all $q \in V$ the map $\phi_q = \phi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

2.2 Definition. Let (E, M, π) be an n -dimensional topological vector bundle over M . A collection

$\beta = \{(\pi^{-1}(V_\alpha), \phi_\alpha) \mid \alpha \in I\}$
of bundle charts is called a bundle atlas for (E, M, π) if $M = \cup_\alpha V_\alpha$.
For each pair (α, β) there exists a function $A_{\alpha, \beta} : V_\alpha \cap V_\beta \rightarrow GL(\mathbb{R}^n)$ such that the corresponding continuous map
 $\phi_\alpha \circ \phi_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^n \rightarrow (V_\alpha \cap V_\beta) \times \mathbb{R}^n$
is given by

$$(p, v) \rightarrow (p, (A_{\alpha, \beta}(p))(v))$$

The elements of $\{A_{\alpha, \beta} \mid \alpha, \beta \in I\}$ are called the transition maps of the bundle atlas β .

3. Example of Riemannian Metrics

The most basic example of a Riemannian metrics is the Euclidean metrics on \mathbb{R}^n .

The Euclidean metric on \mathbb{R}^n in Cartesian coordinates take the form

$$g_{Eucl} = \delta_{ij} dx_i dx_j = (dx_1)^2 + \dots + (dx_n)^2$$

The following proposition gives rise to memory other examples.

3.1 Proposition (Pullback of Metrics by an Immersion)

Let (P, g) be a Riemannian manifold Q be a smooth manifold and let $f: Q \rightarrow P$ be an immersion (a smooth map such that $f^*p : T_pQ \rightarrow T_{f(p)}P$ is injective for all $p \in Q$). The pullback $f^*g \in \Gamma(T^2Q)$ defines a metric on Q .

$$(f^*g)_p(u, v) = g_{f(p)}(f^*p^u, f^*p^v)$$

for all $u, v \in T_pQ$ and all $p \in Q$. Clearly $(f^*g)_p$ is symmetric. Since f^*p is injective, if $f^*pu = 0$ then it follows that $u=0$ is positive definite.

- If Q is a sub-manifold of a Riemannian manifolds (P, g) , the inclusion map $i : Q \rightarrow P$ defines a metric on Q , i^*g , called the induced metric.

In particula, the Euclidean metric induces a metric on every sub-manifold of \mathbb{R}^n .

- Consider the n sphere, $S^n = \{(x^1)^2 + \dots + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$ and the inclusion map $i : S^n \rightarrow \mathbb{R}^{n+1}$.

The induced Euclidean metric i^*g_{Eucl} on S^n is called the round metric (standard metric) on S^n .

4. Riemannian Metrics

We are now in a position to define a Riemannian metric, and some immediate geometric nations such an object gives rise to. First, recall the definition of an inner product.

4.1 Definition (Inner product)

An inner product on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ which is

- Symmetric : $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$
- Bilinear : $\langle a u + b v, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$
- Positive definite : $\langle u, u \rangle > 0$ for all $u \neq 0$

4.2 Example (Euclidean dot product)

The Euclidean dot product is an inner product on \mathbb{R}^n . A Riemannian metric on a manifolds M associates to each point $p \in M$, in a smooth way on inner product on the tangent space $T_p M$.

5. Riemannian Metrics

A Riemannian metric on a manifolds M is a $(0,2)$ tensor field $g \in \Gamma(T^2_0 M)$ which is

- Symmetric : $g_p(X_p, Y_p) = g_p(Y_p, X_p)$ for all $X_p, Y_p \in T_p M$ and all $p \in M$
- Position definite $g_p(X_p, Y_p) > 0$ all $X_p \in T_p M$ with $X_p \neq 0$ and all $p \in M$

6. Riemannian Manifold

A pair (M, g) , where M is smooth manifold and g is a Riemannian metric on M , is called a Riemannian manifold, More generally, one can define a semi-Riemannian metric by relaxing the positive definite condition.

6.1 Definition (Semi Riemannian Metrics)

A semi-Riemannian metric or pseudo-Riemannian metric on a manifold M is a $(0,2)$ tensor field $g \in \Gamma(T^2_0 M)$ which is

- Symmetric : $g_p(X_p, Y_p) = g_p(Y_p, X_p)$ for all $X_p, Y_p \in T_p M$ and all $p \in M$
- Non degenerate : for all $p \in M$, if $X_p \in T_p M$ is such that $g(X_p, Y_p) = 0$ for all $Y_p \in T_p M$, then $X_p = 0$

7. The Levi-Civita Connection

We given an example of a connection in the normal bundle of a sub-manifold of a Riemannian manifold and study its properties.

On the n -dimensional real vector space \mathbb{R}^m we have the well-known differential operator $\partial : C^\infty(T \mathbb{R}^m) \times C^\infty(T$

\mathbb{R}^m \rightarrow $C^\infty(T)$ \mathbb{R}^m
 mapping a pair of vector fields X, Y on \mathbb{R}^m to the directional derivative $\partial_X Y$ of Y in the direction of X given by
 $(\partial_X Y)(x) = \lim_{t \rightarrow 0} \frac{Y(x+tX(x)) - Y(x)}{t}$

The most fundamental properties of the operator ∂ are expressed by the following. If $\lambda, \mu \in \mathbb{R}, f, g \in C^\infty(\mathbb{R}^m)$ and $X, Y, Z \in C^\infty(T)$ then

- (i) $\partial_X(\lambda \cdot Y + \mu \cdot Z) = \lambda \cdot \partial_X Y + \mu \cdot \partial_X Z,$
- (ii) $\partial_X(f \cdot Y) = \partial_X(f) \cdot Y + f \cdot \partial_X Y,$
- (iii) $\partial_{(f \cdot X + g \cdot Y)} Z = f \cdot \partial_X Z + g \cdot \partial_Y Z$

Further, well known properties of the differential operator ∂ are given by the next result .

7.1 Proposition

Let the real vector space \mathbb{R}^m be equipped with the standard Euclidean metric $\langle \cdot, \cdot \rangle$ and $X, Y, Z \in C^\infty(T, \mathbb{R}^m)$ be smooth vector fields .Then

- (iv) $\partial_X Y - \partial_Y X = [X, Y]$
- (v) $\partial_X \langle Y, Z \rangle = \langle \partial_X Y, Z \rangle + \langle Y, \partial_X Z \rangle$

We shall generalize the operator ∂ on the Euclidean space to any Riemannian manifold (M, g) . first we define concept of a connection in a smooth vector bundle .

7.2 Definition

Let (E, M, π) be a smooth vector bundle over M . A connection on (E, M, π) is a map $\nabla : C^\infty(TM) \times C^\infty(E) \rightarrow C^\infty(E)$ such that

- (i) $\nabla_X(\lambda \cdot \vartheta + \mu \cdot \omega) = \lambda \cdot \nabla_X \vartheta + \mu \cdot \nabla_X \omega$
- (ii) $\nabla_X(f \cdot \vartheta) = X(f) \cdot \vartheta + f \cdot \nabla_X \vartheta,$
- (ii) $\nabla(f \cdot X + g \cdot Y) \vartheta = f \cdot \nabla_X \vartheta + g \cdot \nabla_Y \vartheta$

for all $\lambda, \mu \in \mathbb{R}, X, Y \in C^\infty(TM), \vartheta, \omega \in C^\infty(E)$ and $f, g \in C^\infty(M)$.

A section $\vartheta \in C^\infty(E)$ is said to be parallel with respect to the connection ∇ if $\nabla_X \vartheta = 0$ for all $X \in C^\infty(TM)$

7.3 Definition (Compatible Connection)

An affine connection ∇ on a Riemannian manifold (M, g) is called compatible with g if g is parallel with respect to ∇ i.e. if $\nabla g = 0$

7.4 Definition (Torsion tensor field)

Given a connection ∇ on M , define the torsion tensor to be the map $r : X(M) \times X(M) \rightarrow X(M)$ defined by

$r(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

7.5 Definition (Torsion free connection)

An affine Connection ∇ on a manifold M is called symmetric, or torsion free if its torsion tensor ζ identically vanishes i.e. if

$\nabla_X Y - \nabla_Y X = [X, Y]$ For all $X, Y \in X(M)$

8. Theorem (Existence and uniqueness of a compatible Torsion free connection)

Given a Riemannian manifold (M, g) , there exists a unique affine connection ∇ on M in which is torsion free and compatible with g . This affine connection ∇ moreover satisfies

$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) + g(Z, [X, Y])$ -----(1)

Proof- The proof proceeds by showing that any such connection must satisfy equation 1 and then checking that equation 1 indeed defines a torsion free compatible connection. Indeed , suppose ∇ is such a connection . then

Since ∇ is compatible with g .

$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, \nabla_X Z)$
 and since ∇ is torsion free,

$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, [X, Z]) - g(Y, \nabla_Z X)$ -----(2)

Similarly, interchanging the roles of $X, Y,$ and $Z,$

$g(\nabla_Y Z, X) = Y(g(X, Z)) + g(Z, [X, Y]) - g(Z, \nabla_X Y)$ -----(3)

and $g(\nabla_Z X, Y) = Z(g(X, Y)) - g(X, [Y, Z]) - g(X, \nabla_Y Z)$ -----(4)

adding (2) and (3) and subtracting (4) then yields (1).

The uniqueness of such a connection is then evident. existence, ex it is left as an excises to check that (1) indeed defines a torsion free compatible connection .

Example

To check that (1) defines a connection, let $\{e_i\}$ be a local orthonormal frame, set

$$\nabla_x y = \sum_{k=1}^n g(\nabla_x y, e_k) e_k$$

And replace $g(\nabla_x y, e_k)$ using (1). Check that ∇ satisfies the proportion of connection and affine connection and then check that ∇ is torsion free and compatible with g .

9. An overview of the Riemannian Metrics on spaces of curves using the Hamiltonian Approach

The Hamiltonian approach also provides a mechanism for converting symmetries of underlying Riemannian manifolds into conserved quantities, the momenta. We first need to be quite formal consists of smooth vector fields along immersions whereas the latter is comprised of 1-currents along immersions. Because of this we work on the tangent bundle and we pull back the symplectic form from the cotangent bundle to T Immersion (S^1, \mathbb{R}^2) . We use the basics of symplectic geometry and momentum mappings on cotangent bundles in infinite dimensions.

10. Metrics and Momenta on the group of Diffeomorphisms

Very similar things happen when we consider metrics on the group Diff (\mathbb{R}^2) . The tangent space to Diff (\mathbb{R}^2) at the identity is the vector space of vector fields $X(\mathbb{R}^2)$ on \mathbb{R}^2 and we can identify T Diff (\mathbb{R}^2) with the product Diff $(\mathbb{R}^2) \times X(\mathbb{R}^2)$ using right multiplication in the group to identify the tangent at a point ϕ with that at the identity. The definition of this product decomposition means that right multiplication by ψ carries (ϕ, X) to $(\phi\psi, X)$. As usual, suppose that conjugation $\phi \rightarrow \psi\phi\psi^{-1}$ has the derivative at the identity given by the linear operator Ad_ψ on the Lie algebra $X(\mathbb{R}^2)$. It is easy to calculate the explicit formula for Ad ;

$$Ad_\psi(X) = (D\psi \cdot X) \circ \psi^{-1}$$

Then left multiplication by ψ on Diff $(\mathbb{R}^2) \times X(\mathbb{R}^2)$ is given by $(\phi, X) \rightarrow (\psi\phi, Ad_\psi(X))$. We now want to carry over the ideas of section given above replacing the space Immersion (S^1, \mathbb{R}^2) by Diff (\mathbb{R}^2) and the group action there by the right action of Diff (\mathbb{R}^2) on itself. The Lie algebra g is therefore $X(\mathbb{R}^2)$ and the fundamental vector field $\tau_x(c)$ is now the vector field with value

$$\tau_x(\phi) = \delta t \text{Io}(\phi \xrightarrow{\text{yields}} \phi \circ \exp(tx) \circ \phi^{-1} - 1) = Ad_\phi(x)$$

at the point ϕ . We now assume we have a positive definite inner product $G(X, Y)$ on the Lie algebra $X(\mathbb{R}^2)$ and that we use right translation to extend it to a Riemannian metric on the full group Diff (\mathbb{R}^2) . The metric being, by definition, invariant under the right group action, we have the setting for momentum. The theory of the last section tells us to define the momentum mapping by:

$$jx(\phi, Y) = G(\tau_x(\phi), Y)$$

Noether's theorem tells us that if $\phi(t)$ is a geodesic in Diff (\mathbb{R}^2) for this metric, then this momentum will be constant along the lift of this geodesic to the tangent space. The lift of $\phi(t)$, in the product decomposition of the tangent space is the curve

$$t \rightarrow (\phi(t), \partial t(\phi) \circ \phi^{-1}(t))$$

hence the theorem tells us that;

$$G(Ad_{\phi(t)}(x), \partial t(\phi) \circ \phi^{-1}(t)) = \text{Constant}$$

for all x . If we further assume that Ad has an adjoint with respect to G :

$$G(Ad_\phi(x), Y) = G(X, Ad_{\phi^{-1}}^*(Y))$$

then this invariance of momentum simplifies to:

$$Ad_{\phi(t)}^*(\partial t(\phi) \circ \phi^{-1}(t)) = \text{constant}$$

This is a very strong invariance and it encodes an integrand form of the geodesic equations for the group.

11. Weak Riemannian Manifolds

A Riemannian metric a Hilbert on Frechet manifold is a smooth $(0,2)$ tensor that induces a bounded, positive - definite scalar product on each tangent space. There are two kinds of Riemannian metric an infinite-dimensional manifolds: If the tangent spaces are complete with respect to the scalar product induced by the metric, it is called strong. Otherwise, it is called weak. As we will see, weak metrics are significantly more technically challenging than strong metrics. However, in the case of a proper Frechet space (where the topology does not come from any single norm), only weak Riemannian metrics are possible. This, among other considerations, comments their importance in global analysis and the value of their Study.

The subtle but important distinction between weak and strong Riemannian metric leads to a vast gulf in their properties. For a strong Riemannian metric, one can reproduce many of the important results from finite-dimensional

Riemannian geometry. For example the Levi-Civita connections, geodesics, and the exponential mapping on its. A strong Riemannian metric induces a distance function that gives a metric space structure on the manifolds, though some can be directly shown for many important examples.

In this section, we will give some basic results on the distance function of a weak Riemannian manifold. We have not found these results formally recorded anywhere, though they may be known to experts in the field.

Our approach essentially follows that of [1], which treats the case of strong Riemannian manifolds. We have made the necessary adjustments to the results and proofs so that they hold in the weak case.

For the remainder of this section, let (N, Y) be a Riemannian Frechet manifold. Just as in the case of finite dimensional Riemannian manifolds, we can use Y to define a distance between points of N by taking the infimum of lengths of paths. It is then clear that this distance function is a pseudo-metric, but it may fail to be positive definite. The problem in showing positive-definiteness on a weak Riemannian manifold is that the exponential mapping and its inverse need not be defined on an open Y -or-d ball, respectively. On the other hand this is a vital ingredient in the proof for strong Riemannian metrics.

12. The Hamiltonian Vector Field Mappings

Here we compute the Hamiltonian vector field $\text{grad}^\omega(f)$ associated to a smooth function f on the tangent space $T \text{Immersion}(S^1, \mathbb{R}^2)$, that is $f \in C^\infty(\text{Immersion}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2))$ assuming that it has (smooth G -gradients in both factors. Using the explicit formulas, we have;

$$\begin{aligned} \mathcal{G}_{(c,h)}(\text{grad}^\omega(f)(c,h),(k,l)) &= \mathcal{G}_{(c,h)}((\text{grad}_1^\omega(c,h), \text{grad}_2^\omega(c,h)), (k,l)) = G_c(k, H_c(h, \text{grad}_1^\omega(f)(c,h))) - \\ & G_c(K_c(\text{grad}_1^\omega(f)(c,h), h), k) + G_c(l, \text{grad}_1^\omega(f)(c,h)) - G_c(\text{grad}_2^\omega(f)(c,h), k). \end{aligned}$$

On the other hand, by the definition of the \mathcal{G} -gradient we have

$$\mathcal{G}_{(c,h)}(\text{grad}^\omega(f)(c,h),(k,l)) = df(c,h)(k,l) = D_{(c,k)}f(c,h) + D_{(h,l)}f(c,h) = G_c(\text{grad}_1^G(f)(c,h), k) + G_c(\text{grad}_2^G(f)(c,h), l)$$

And we get the expression of the Hamiltonian vector field;

$$\text{grad}_1^\omega(f)(c,h) = \text{grad}_2^\omega(f)(c,h),$$

$$\text{grad}_2^\omega(f)(c,h) = -\text{grad}_1^\omega(f)(c,h) + H_c(h, \text{grad}_2^\omega(f)(c,h)) - K_c(\text{grad}_2^\omega(f)(c,h), h).$$

Note that for a smooth function f on $T \text{Immersion}(S^1, \mathbb{R}^2)$ the \mathcal{G} -gradient exists if and only if both G -gradient exists.

13. The Geodesic Equation

The geodesic flow is defined by a vector field on $T \text{Immersion}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2) \xrightarrow{\text{yields}} IR$

The two partial G -gradient are:

$$G_c(\text{grad}_2^G(E)(c,h), l) = d_2 E(c,h)(l) = G_c(h, l),$$

$$\text{Grad}_2^G(E)(c,h) = h,$$

$$G_c(\text{grad}_1^G(E)(c,h), k) = d_1 E(c,h)(k) = 1/2 D_{(c,k)} G_c(h,h) = 1/2 G_c(k, H_c(h,h)).$$

$$\text{Grad}_1^G(E)(c,h) = 1/2 H_c(h,h).$$

Thus geodesic vector field is

$$\text{grad}_1^\omega(E)(c,h) = h,$$

$$\text{grad}_2^\omega(E)(c,h) = 1/2 H_c(h,h) - K_c(h,h)$$

and the geodesic equation becomes :

$$c_t = h,$$

$$h_t = 1/2 H_c(h,h) - K_c(h,h) \quad \text{or} \quad c_{tt} = 1/2 H_c(c_t, c_t) - K_c(c_t, c_t)$$

This is nothing but the usual formula for the geodesic flow using the Christoffel symbols expanded out using the first derivatives of the metric tensor.

14. The Momentum Mappings for a G -Isometric Group Action

We consider now a (possibly infinite-dimensional regular) Lie group with Lie algebra \mathfrak{g} with a right action $g \xrightarrow{\text{yields}} \mathfrak{r}^g$ by isometric on $\text{Immersion}(S^1, \mathbb{R}^2)$. If $\mathfrak{X}(\text{Immersion}(S^1, \mathbb{R}^2))$ denotes the set of vector fields on $\text{Immersion}(S^1, \mathbb{R}^2)$, we can

specify this action by the fundamental vector field mapping $\zeta: \mathfrak{g} \xrightarrow{\text{yields}} \mathfrak{X}(\text{Immersion}(S^1, \mathbb{R}^2))$, which will be bounded Lie algebra homomorphism. The fundamental vector field $\zeta_X, X \in \mathfrak{g}$ is the infinitesimal action in the sense:

$$\zeta_X(c) = \partial_t / \partial r^{\exp(tX)}(c).$$

We also consider the tangent prolongation of this action on $T \text{Immersion}(S^1, \mathbb{R}^2)$ where fundamental Vector field is given by

$$\zeta_X^{T \text{Immersion}}(c,h) \xrightarrow{\text{yields}} (c,h; \zeta_X(c), D_{(c,h)}(\zeta_X)(c) =: \zeta_X'(c,h)).$$

The basic assumption is that the action is by isometries,

$$G_c(h,k) = ((r^g)^* G)_c(h,k) = G_{rg(c)}(T_c(r^g)h, T_c(r^g)k).$$

Differentiating this equation at $g=e$ in the direction $X \in \mathfrak{g}$ we get

$$0 = D_{(c, \zeta_X(c))} G_c(h,k) + G_c(\zeta_X'(c,h), k) + G_c(h, \zeta_X'(c,k)).$$

The key to the Hamiltonian approach is to define the group action by Hamiltonian flows. To do this ,we define the momentum map $j: \mathfrak{g} \xrightarrow{yields} C_G^\infty(T \text{ Immersion } (S^1, \mathbb{R}^2), \mathbb{R})$ by:

$$j_X(c, h) = G_c(\zeta_X(c), h),$$

Equivalently ,since this map is linear ,it is often written as a map

$$\zeta: T \text{ Immersion } (S^1, \mathbb{R}^2) \xrightarrow{yields} \mathfrak{g}', \quad (\zeta(c, h), X) = j_X(c, h).$$

The main property of the momentum map is that it fits into following commutative diagram and is a homomorphism of Lie algebra:

$$\begin{array}{ccccc} H^0(T \text{ Immersion }) & \xrightarrow{yields} & C_G^\infty(T \text{ Immersion } , \mathbb{R}) & \xrightarrow{yields} & X(T \text{ Immersion } , \mathfrak{G}) & \xrightarrow{yields} & H^1(T \text{ Immersion }) \\ \downarrow \text{yields} & & \downarrow \text{yields} & & \downarrow \text{yields} & & \downarrow \text{yields} \\ \mathfrak{g} & \xrightarrow{yields} & C_G^\infty(T \text{ Immersion } , \mathbb{R}) & & X(T \text{ Immersion } , \omega) & & \end{array}$$

where $X(T \text{ Immersion}, \omega)$ is the space of vector fields on $T \text{ Immersion}$ whose flow leaves ω fixed . We need to check that:

$$\zeta_X(c) = \text{grad}_1^{\circ} (jX)(c, h) = \text{grad}_2^{\circ} (jX)(c, h),$$

$$\zeta'_X(c, h) = \text{grad}_2^{\circ} (jX)(c, h) = -\text{grad}_1^{\circ} (jX)(c, h) + H_c(h, \zeta_X(c)) - K_c(\zeta_X(c), h)$$

The first equation is obvious. To verify the second equation, We take its inner product with some k and use:

$$\begin{aligned} G(k, \text{grad}_1^{\circ} (jX)(c, h)) &= D_{(c, k)} jX(c, h) = D_{(c, k)} G_c(\zeta_X(c), h) + G_c(\zeta'_X(c, k), h) \\ &= G_c(k, H_c(\zeta_X(c), h)) + G_c(\zeta'_X(c, k), h). \end{aligned}$$

Combining this with (2) , the second equation follows. Let us check that it is also a homomorphism of Lie algebras using the Poisson bracket:

$$\begin{aligned} \{j_X, j_Y\}(c, h) &= d_{j_Y}(c, h)(\text{grad}_1^{\circ} (jX)(c, h), \text{grad}_2^{\circ} (jX)(c, h)) \\ &= d_{j_Y}(c, h)(\zeta_X(c), \zeta'_X(c, h)) \\ &= D_{(c, \zeta_X(c))} G_c(\zeta_Y(c), h) + G_c(\zeta'_Y(c), \zeta_X(c), h) + G_c(\zeta_Y(c), \zeta'_X(c, h)) \\ &= G_c(\zeta'_Y(c), \zeta_X(c)) - \zeta'_X(c, \zeta_Y(c), h) \quad \text{by 2} \\ &= G_c([\zeta_X, \zeta_Y](c), h) = G_c(\zeta_{[X, Y]}(c), h) = j_{[X, Y]}(c). \end{aligned}$$

Note also that ζ is equivalent for the group action ,by the following arguments: For g in the Lie group let r^g be the right action on $\text{Immersion } (S^1, \mathbb{R}^2)$, then $T(r^g) \circ \zeta_X \circ (r^g)^{-1} = \zeta_{\text{Ad}(g^{-1})X}$. Since r^g is an isometric the mapping $T(r^g)$ is a symplectomorphism for ω , thus grad° is equivalent . Thus $jX \circ T(r^g) = j_{\text{Ad}(g)X}$ plus a possible constant which we can rule out since $j_X(c, h)$ is linear in h .

By Emmy Noether's theorem, along any geodesic $t \xrightarrow{yields} c(t, \cdot)$ this momentum mapping is constant, thus for any $X \in \mathfrak{G}$ we have

$$(\zeta(c, c_t), X) = j_X(c, c_t) = G_c(\zeta_X(c), c_t) \text{ is constant in } t.$$

We can apply this construction to the following group actions on $\text{Immersion } (S^1, \mathbb{R}^2)$.

The smooth right action of the group $\text{Diff}(S^1)$ on $\text{Immersion } (S^1, \mathbb{R}^2)$, given by composition from the right $: c \xrightarrow{yields} c \circ \phi$ for $\phi \in \text{Diff}(S^1)$. For $X \in X(S^1)$ the fundamental vector field is then given by

$$\zeta_X^{\text{Diff}}(c) = \zeta_X(c) = \partial_t / 0(c \circ F_t^X) = c_{\theta} X$$

where F_t^X denotes the flow of X . The reparametrization momentum, for any vector field X on S^1 is thus:

$$j_X(c, h) = G_c(c_{\theta}, X, h).$$

Assuming the metric is reparametrization invariant, it follows that on any geodesic $c(\theta, t)$, the expression $G_c(c_{\theta}, X, c_t)$ is constant for all X .

The left action of the Euclidean motion group $M(2) = \mathbb{R}^2 \times S o(2)$ on $\text{Imm}(S^1, \mathbb{R}^2)$ given by $c \xrightarrow{yields} e^{aJ} c + B$ for $(B, e^{aJ}) \in \mathbb{R}^2 \times S o(2)$. The fundamental vector field mapping is

$$\zeta_{(B, a)}(c) = a J c + B.$$

The linear momentum is thus $G_c(B, h)$, $B \in \mathbb{R}^2$ and if the metric is translation invariant, $G_c(B, c_t)$ will be constant along geodesics. The angular momentum is similarly $G_c(J c, h)$ and if the metric $G_c(J c, c_t)$ will be constant along geodesic.

The action of the scaling group of \mathbb{R} given by $c \xrightarrow{yields} e^r c$, with fundamental vector field $\zeta_a(c) = a \cdot c$. If the metric is scale invariant, then the scaling momentum $G_c(c, c_t)$ will also be invariant along geodesics.

15. Conclusions

The Levi-Civita connection, named after mathematicians Tullio Levi -Civita and Gregorio Ricci-Curvature is a fundamental concept in differential geometry and the theory of connections on differentiable manifolds. It allows for the definition of parallel transport and the notion of covariant derivatives, which are essential in various branches of mathematical and theoretical physics, particularly in general relativity. In conclusion, the Levi-Civita connection provides a key framework for understanding curvature and geometric properties of spaces with smooth structures.

Levi- Civita connection is a fundamental concept within Riemannian geometry, as it plays a central role in defining geodesics, curvature and parallel transport on Riemannian manifolds.

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