BANACH SPACE VALUED SEQUENCE SPACE

\[ l_M(X, p, u) \]

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Abstract- In this paper, we introduce the Banach space valued sequence space \( l_M(X, p, u) \) and examine various algebraic and topological properties of it. Finally we introduce a subspace of \( l_M(X, p, u) \) and investigate some topological properties of it. Our results generalize and unify the corresponding earlier results of Kamthan and Gupta [3], Ahmad and Bataineh [1].

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1. Introduction

Lindenstrauss and Tzafriri[6] used the idea of an Orlicz function \( M \) to construct the sequence space \( l_M \) of all sequences of scalars \( (x_k) \) such that \( \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \) for some \( \rho > 0 \). The space \( l_M \) equipped with the norm

\[ \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} \]

is a BK space [3, p. 300] usually called an Orlicz sequence space. The space \( l_M \) is closely related to the space \( l_p \), which is an Orlicz sequence space with \( M(x) = x^p, 1 \leq p < \infty \).

We recall [3, 6] that an Orlicz function \( M \) is a function from \([0, \infty)\) to \([0, \infty)\) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \) for all \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). Note that an Orlicz function is always unbounded.

An Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)-condition for all values of \( u \) if there exists a constant \( K > 0 \) such that \( M(2u) \leq KM(u), \ u \geq 0 \). It is easy to see that always \( K > 2[4] \). A simple example of an Orlicz function which satisfies the \( \Delta_2 \)-condition for all values of \( u \) is given by \( M(u) = a|u|^\alpha (\alpha > 1) \), since \( M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u) \). The Orlicz function \( M(u) = e^{\|u\|} - |u| - 1 \) does not satisfy the \( \Delta_2 \)-condition.

The \( \Delta_2 \)-condition is equivalent to the inequality \( M(lu) \leq K(l)M(u) \) which holds for all values of \( u \), where \( l \) can be any number greater than unity.

An Orlicz function \( M \) can always be represented in the following integral form

\[ M(x) = \int_{0}^{x} p(t)dt \]

where \( p \) known as the kernel of \( M \), is right differentiable for \( t \geq 0, p(0) = 0, p(t) > 0 \) for \( t > 0, p \) is non-decreasing and \( p(t) \to \infty \) as \( t \to \infty \).

Before proceeding with the main results we recall [7; second edition] some terminology and notations.

A paranormed space \( X = (X, g) \) is a topological linear space in which the topology is given by a paranorm \( g \); a real subadditive function on \( X \) such that \( g(\theta) = 0, g(\xi) = g(-\xi) \) and such that the scalar multiplication is continuous. In the above, \( \theta \) is the zero in the complex linear space \( X \) and continuity of multiplication means that \( \lambda_n \to \lambda, x_n \to x \) (i.e., \( g(\lambda_n x_n - \lambda x) \to 0 \)) imply \( \lambda_n x_n \to \lambda x \) (i.e., \( g(\lambda_n x_n - \lambda x) \to 0 \)), for scalars \( \lambda \) and vectors \( x \).

A paranorm for which \( g(x) = 0 \) implies \( x = \theta \) is called total paranorm.

A Frechet space is a complete metric linear space, or equivalently a complete totally paranormed space.

Let \( w \) denote the space of all complex sequences \( x = (x_n) \). Let \( X \) be a linear subspace of \( w \) such that \( X \) is a Frechet space with continious coordinate projections. Then we say that \( X \) is an FK space, or a Frechet Koordinat space. If the metric of an FK space \( X \) is given by a complete norm then we say that \( X \) is a BK space, i.e. a Banach Koordinat space.
A sequence \((b_k)\) of elements of a paranormed space \((X, g)\) is called a Schauder basis for \(X\) if and only if, for each \(x \in X\), there exists a unique sequence \((\lambda_k)\) of scalars such that \(x = \sum_{k=1}^{\infty} \lambda_k b_k\), i.e., such that \(g(x - \sum_{k=1}^{n} \lambda_k b_k) \to 0 (n \to \infty)\).

An FK space \(X\) has AK, or has the AK property, if \((e_k)\), the sequence of unit vectors, is a Schauder basis for \(X\). In effect, this means that for each \(x = (x_k) \in X\) we have \((x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \sum_{k=1}^{n} x_k e_k \to x (n \to \infty)\), where the convergence is in the metric of \(X\).

Let \((X, \|\cdot\|)\) be a Banach space over the complex field \(\mathbb{C}\). Denote by \(w(X)\) the space of all \(X\)-valued sequences. Let \(M\) be an Orlicz function, \(u = (u_k)\) be an arbitrary sequence of scalars such that \(u_k \neq 0 (k = 1, 2, \ldots)\) and \(p = (p_k)\) be a bounded sequence of positive real numbers.

We now introduce the Banach space valued sequence space \(l_M(X, p, u)\) using an Orlicz function \(M\) as follows:

\[ l_M(X, p, u) = \{x \in w(X) : \sum_{k=1}^{\infty} \left( M\left( \frac{|u_k x_k|^p}{\rho} \right) \right)^{\frac{1}{p}} < \infty \text{ for some } \rho > 0 \} \]

Some well-known spaces are obtained by specializing \(X, M, p\) and \(u\).

(i) If \(X = \mathbb{C}, p_k = u_k = 1\) for all \(k\), then \(l_M(X, p, u) = l_M\) (Lindenstrauss and Tzafriri [6]).

(ii) If \(X = \mathbb{C}, u_k = 1\) for all \(k\), then \(l_M(X, p, u) = l_M(p)\) (Parashar and Choudhary [8]).

(iii) If \(X = \mathbb{C}\), then \(l_M(X, p, u) = l_M(p, u)\) (Ahmad and Bataineh [1]).

(iv) If \(M(x) = x, u_k = 1\) for all \(k\) and \(p_k = p(1 \leq p < \infty)\) for all \(k\), then \(l_M(X, p, u) = l_p(X)\) (Leonard [5]).

We denote \(l_M(X, p, u)\) as \(l_M(X, p)\) when \(u_k = 1\) for all \(k\).

In §2, we propose to study various algebraic and topological properties of the sequence space \(l_M(X, p, u)\). In §3, certain inclusion relations between \(l_M(X, p, u)\) space have been established. In §4, some information on multipliers for \(l_M(X, p, u)\) is given. In §5, a subspace of \(l_M(X, p, u)\) has been introduced and some topological properties of it has been discussed.

The following inequalities (see, e.g., [7; first edition, p. 190]) are needed throughout the paper.

Let \(p = (p_k)\) be a bounded sequence of positive real numbers. If \(H = \sup_k p_k\), then for any complex \(a_k\) and \(b_k\),

\[(1.1) \quad |a_k + b_k|^{p_k} \leq C (|a_k|^{p_k} + |b_k|^{p_k}),\]

where \(C = \max(1, 2^H - 1)\). Also for any complex \(\lambda\),

\[(1.2) \quad |\lambda|^{p_k} \leq \max(1, |\lambda|^H)\).

2. Linear topological structure of \(l_M(X, p, u)\) spaces

**Theorem 2.1.** For any Orlicz function \(M\), \(l_M(X, p, u)\) is a linear space over the complex field \(\mathbb{C}\). The proof is a routine verification by using standard techniques and hence is omitted.

**Theorem 2.2.** \(l_M(X, p, u)\) is a topological linear space, paranormed by

\[(2.1) \quad g(x) = \inf \left\{ \rho^{p/n} G \left( \sum_{k=1}^{\infty} \left[ M\left( \frac{|u_k x_k|}{\rho} \right) \right]^{\frac{1}{p}} \right)^{\frac{1}{n}} \leq 1 \right\}\]

where \(G = \max(1, \sup_k p_k)\).

The proof uses ideas similar to those used (e.g.) in [8, p. 421] and the fact that every paranormed space is a topological linear space [9, p. 37].

**Theorem 2.3.** Let \(1 \leq p_k < \infty\), then \(l_M(X, p, u)\) is a Frechet space paranormed by (2.1).

**Proof.** Let \((x^i)\) be a Cauchy sequence in \(l_M(X, p, u)\). Let \(r, u_0\) and \(x_0\) be fixed. Then for each \(\frac{\epsilon}{ru_0x_0} > 0\) there exists a positive integer \(N\) such that

\[ g(x^i - x^j) < \frac{\epsilon}{ru_0x_0}, \text{ for all } i, j \geq N. \]

Using definition of paranorm, we get
\[
\left( \sum_{k=1}^\infty M \left( \frac{\|u_k^j x_k^i - u_k^j x_i^i\|}{g(x^i - x^j)} \right) \right)^{\frac{1}{p_k}} \leq 1, \text{ for all } i, j \geq N.
\]

Thus
\[
\sum_{k=1}^\infty M \left( \frac{\|u_k^j x_k^i - u_k^j x_i^i\|}{g(x^i - x^j)} \right)^{p_k} \leq 1, \text{ for all } i, j \geq N.
\]

Since \(1 \leq p_k < \infty\), it follows that
\[
M \left( \frac{\|u_k^j x_k^i - u_k^j x_i^i\|}{g(x^i - x^j)} \right) \leq 1,
\]
for each \(k \geq 1\) and for all \(i, j \geq N\). Hence one can find \(r > 0\) with \(\|u_0 x_0\| / 2 \geq 1\), where \(p\) is the kernel associated with \(M\), such that
\[
M \left( \frac{\|u_k^j x_k^i - u_k^j x_i^i\|}{g(x^i - x^j)} \right) \leq \frac{\|u_0 x_0\|}{2} r p \frac{\|u_0 x_0\|}{2}.
\]

Using the integral representation of Orlicz function \(M\), we get
\[
\|u_k^j x_k^i - u_k^j x_i^i\| \leq \frac{r \|u_0 x_0\|}{2} g(x^i - x^j)
\]
\(< \frac{\epsilon}{2}\), for all \(i, j \geq N\).

Hence \((u^j x^j)\) is a Cauchy sequence in \(X\) which implies that \((x^j)\) is Cauchy in \(X\) since \(u\) is an arbitrary fixed sequence of parameters such that \(u_k \neq 0\) for each \(k\). Therefore, for each \(\epsilon(0 < \epsilon < 1)\), there exists a positive integer \(N\) such that
\[
\|x^i - x^j\| < \epsilon\] for all \(i, j \geq N\).

Now, using continuity of \(M\), we find that
\[
\left( \sum_{k=1}^N M \left( \frac{\|u_k^j (x_k^i - \lim_{j \to \infty} x_k^i)\|}{\rho} \right) \right)^{\frac{1}{p_k}} \leq 1, \text{ for all } i \geq N.
\]

Thus
\[
\left( \sum_{k=1}^N M \left( \frac{\|u_k^j (x_k^i - x_k^i)\|}{\rho} \right) \right)^{\frac{1}{p_k}} \leq 1, \text{ for all } i \geq N.
\]

Since \(N\) is arbitrary, by taking infimum of such \(\rho\)'s we get
\[
\inf \left\{ \rho^{p_n} : \left( \sum_{k=1}^\infty M \left( \frac{\|u_k^j (x_k^i - x_k^i)\|}{\rho} \right) \right)^{\frac{1}{p_k}} \leq 1 \right\} \text{ for all } i \geq N.
\]

Hence \(g(x^i - x) < \epsilon\) for all \(i \geq N\). That is to say that \((x^i)\) converges to \(x\) in the paranorm of \(l_M(X, p, u)\). Since \((x^i) \in l_M(X, p, u)\) and \(M\) is continuous, it follows that \(x \in l_M(X, p, u)\).

**Corollary 2.4.** If \(p\) is a constant sequence, then \(l_M(X, p, u)\) is a Banach space for \(p \geq 1\) and a complete \(p\)-normed space for \(p < 1\).

**Definition 2.5[2]** A linear subspace \(Y\) of \(w(X)\) is a generalized FK space (resp. a generalized BK space) if \(Y\) is a Fre’chet space (resp. a Banach space) with continuous coordinate projections.

In case \(X = \mathbb{C}\), then \(Y\) becomes an FK space (resp. a BK space).

**Theorem 2.6.** Let \(1 \leq p_k < \infty\), then \(l_M(X, p)\) is a generalized FK space paranormed by \((2.1)\).

**Proof.** In view of Theorem 2.3, it is sufficient to show that the coordinate functionals \(P_i; l_M(X, p) \to X\), where \(P_i(x) = x_i\) are continuous.

For \(\epsilon > 0\) let \(\delta > 0\) be such that \(0 < \delta < 1\) and \(\delta \leq M(\epsilon)\). Let \(g(x) < \delta\) so that \(\sum_{k=1}^\infty M \left( \frac{\|x_k\|}{g(x)} \right)^{p_k} \leq 1\)

This implies that \(\sum_{k=1}^\infty M \left( \frac{\|x_k\|}{\delta} \right)^{p_k} \leq 1\)

and so \(M \left( \frac{\|x_k\|}{\delta} \right)^{p_k} \leq 1\) for each \(k \geq 1\).
As \( 1 \leq p_k < \infty \), so \( M \left( \frac{\|x_k\|}{\delta} \right) \leq 1 \) for each \( k \geq 1 \).

Since \( 0 < \delta < 1 \) and \( M \) is convex \( \frac{1}{\delta} M(\|x_k\|) \leq M \left( \frac{\|x_k\|}{\delta} \right) \leq 1 \) which implies that \( M(\|x_k\|) \leq \delta \leq M(\epsilon) \).

Since \( M \) is non-decreasing, we have \( \|x_k\| < \epsilon \) for each \( k \geq 1 \) and hence \( \|x_k\| < \epsilon \) for each \( k \). Thus the coordinate functionals are continuous and this completes the proof of the theorem.

**Corollary 2.7.** If \( p \) is a constant sequence and \( p \geq 1 \), then \( l_M(X, p) \) is a generalized BK space.

3. Inclusion between \( l_M(X, p, u) \) spaces

We now investigate some inclusion relations between \( l_M(X, p, u) \) spaces.

**Theorem 3.1.** If \( p = (p_k) \) and \( q = (q_k) \) are bounded sequences of positive real numbers with \( 0 < p_k \leq q_k < \infty \) for each \( k \), then for any Orlicz function \( M \), \( l_M(X, p, u) \subseteq l_M(X, q, u) \).

**Proof.** Let \( x \in l_M(X, p, u) \). Then there exists some \( \rho > 0 \) such that \( \sum_{k=1}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^p_k < \infty \). This implies that \( M \left( \frac{\|u_k x_k\|}{\rho} \right) \leq 1 \) for sufficiently large values of \( k \), say \( k \geq n_0 \) for some fixed \( n_0 \in N \). Since \( M \) is non-decreasing and \( p_k \leq q_k \), we have

\[
\sum_{k=n_0}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^q_k \leq \sum_{k=n_0}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^p_k < \infty.
\]

This shows that \( x \in l_M(X, q, u) \) and completes the proof.

**Theorem 3.2.** If \( r = (r_k) \) and \( t = (t_k) \) are bounded sequences of positive real numbers with \( 0 < r_k, t_k < \infty \) and if \( p_k = \min(r_k, t_k) \), \( q_k = \max(r_k, t_k) \), then for any Orlicz function \( M \), \( l_M(X, p, u) \cap l_M(X, t, u) \) and \( l_M(X, q, u) = G \), where \( G \) is the subspace of \( w \) generated by \( l_M(X, r, u) \cap l_M(X, t, u) \).

**Proof.** It follows from Theorem 3.1 that \( l_M(X, r, u) \subseteq l_M(X, t, u) \cap l_M(X, t, u) \) and \( G \subseteq l_M(X, q, u) \).

For any complex \( \lambda \), \( |\lambda|^p \leq \max(|\lambda|^r, |\lambda|^s) \), thus \( l_M(X, r, u) \cap l_M(X, t, u) \subseteq l_M(X, p, u) \).

Let \( A = \{k: r_k \geq t_k\} \) and \( B = \{k: r_k < t_k\} \).

If \( x = (x_k) \in l_M(X, q, u) \), we write

\[
y_k = x_k(k \in A) \quad \text{and} \quad y_k = 0(k \in B); \quad \text{and} \quad z_k = x_k(k \in B).
\]

Then since \( x = (x_k) \in l_M(X, q, u) \), there exists some \( \rho > 0 \) such that \( \sum_{k=1}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^q_k < \infty \).

Now,

\[
\sum_{k=1}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^r_k = \sum_{k \in A} + \sum_{k \in B} = \sum_{k \in A} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^q_k < \infty.
\]

and so \( y \in l_M(X, r, u) \subseteq G \).

Similarly, \( z \in l_M(X, t, u) \subseteq G \).

Thus, \( x = y + z \in G \). We have proved that \( l_M(X, q, u) \subseteq G \), which gives the required result.

**Corollary 3.3.** The three conditions \( l_M(X, r, u) \subseteq l_M(X, t, u) \), \( l_M(X, p, u) = l_M(X, r, u) \) and \( l_M(X, t, u) = l_M(X, q, u) \) are equivalent.

**Corollary 3.4.** \( l_M(X, r, u) = l_M(X, t, u) \) if and only if \( l_M(X, p, u) = l_M(X, q, u) \).

4. The Space of Multipliers of \( l_M(X, p, u) \)

For any set \( E \subseteq w(X) \) the space of multipliers of \( E \), denoted by \( S(E) \), is given by \( S(E) = \{a = (a_k) \in w(X): ax = (a_k x_k) \in E \text{ for all } x = (x_k) \in E\} \).

**Theorem 4.1.** For Orlicz function \( M \) which satisfies the \( \Delta_2 \)-condition and Banach algebra \( X \), we have \( l_M(X) \subseteq S[l_M(X, p, u)] \)

where \( l_M(X) = \{a = (a_k) \in w(X): \sup_k \|a_k\| < \infty\} \).

**Proof.** Let \( a = (a_k) \in l_M(X) \), \( T = \sup_k \|a_k\| \) and \( x = (x_k) \in l_M(X, p, u) \). Then \( \sum_{k=1}^{\infty} \left[ M \left( \frac{\|u_k x_k\|}{\rho} \right) \right]^p_k < \infty \) for some \( \rho > 0 \). Since \( M \) satisfies the \( \Delta_2 \)-condition, there exists a constant \( K > 1 \) such that
\[
\sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} \left[ M \left( \frac{\| a_k \|_{U_k} }{\rho} \right) \right]^{p_k} \\
\leq \sum_{k=1}^{\infty} \left[ M \left( (1 + |T|) \frac{\| u_k \|_{L_k} }{\rho} \right) \right]^{p_k} \\
\leq (K(1 + |T|))^{\frac{1}{p}} \sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\rho} \right) \right]^{p_k} < \infty,
\]

where \([T]\) denotes the integer part of \(T\). Hence \(a \in S[l_{m}(X, p, u)]\).

5. A subspace of \(l_{m}(X, p, u)\)

In this section we introduce a subspace of \(l_{m}(X, p, u)\) and investigate some topological properties of it.

We define \(h_{m}(X, p, u)\) by

\[
h_{m}(X, p, u) = \left\{ x = (x_k) \in w(X) : \sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\rho} \right) \right]^{p_k} < \infty \text{ for every } \rho > 0 \right\}.
\]

The space \(h_{m}(X, p, u)\) is clearly a subspace of \(l_{m}(X, p, u)\), and the topology is determined by the paranorm of \(l_{m}(X, p, u)\) given by (2.1).

**Theorem 5.1.** Let \(1 \leq p_k < \infty\). Then \(h_{m}(X, p, u)\) is a Frechet space with the paranorm given by (2.1).

**Proof.** Since \(h_{m}(X, p, u)\) is a subspace of \(l_{m}(X, p, u)\) which is a Frechet space under the paranorm given by (2.1), it is sufficient to show that \(h_{m}(X, p, u)\) is closed in \(l_{m}(X, p, u)\). Therefore, let \((x^i) = (x_k^i)\) be a sequence in \(h_{m}(X, p, u)\) such that \(g(x^i - x) \to 0\) as \(i \to \infty\), where \(x = (x_k) \in l_{m}(X, p, u)\).

To complete the proof we need to show that \(\sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\xi} \right) \right]^{p_k} < \infty\) for every \(\xi > 0\). To \(\xi > 0\) there corresponds an integer \(m\) such that \(g((x^m - x) < \xi/2)\), and so by the convexity of \(M,\)

\[
\sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\xi} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} \left[ \frac{1}{2} M \left( \frac{\| u_k \|_{L_k} }{\xi/2} \right) + \frac{1}{2} M \left( \frac{\| u_k \|_{L_k} }{\xi/2} \right) \right]^{p_k} \\
\leq C \sum_{k=1}^{\infty} \left[ M \left( \frac{\| u_k \|_{L_k} }{\xi/2} \right) \right]^{p_k} < \infty,
\]

where \(C = \max(1, 2^{m-1})\). Thus \(x \in h_{m}(X, p, u)\) which shows that \(h_{m}(X, p, u)\) is complete.

**Corollary 5.2.** Let \(1 \leq p_k < \infty\), then \(h_{m}(X, p, u)\) is a generalized FK space with the paranorm given by (2.1).

The proof follows in view of Theorem 2.6 and Theorem 5.1.

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