On Weighted Juchez Distribution with Properties and its Applications

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Abstract- **In this paper, a new generalization of Juchez distribution called as weighted Juchez distribution has been studied. Its various statistical properties such as moments, survival function, hazard function, order statistics, bonferroni and Lorenz curves has been presented. Its parameters have been estimated by using technique of maximum likelihood estimation. Finally, a new distribution has been demonstrated with real lifetime data set to examine its usefulness.**

Keywords: **weighted distribution, Juchez distribution, survival analysis, order statistics, maximum likelihood estimation.**

1. INTRODUCTION

The concept of weighted distribution is traceable to the work of Fisher (1934) in respect of his studies on how the methods of ascertainment can affect the form of distribution of recorded observations. Later, it was developed in a more general way by Rao (1965) with respect to modeling statistical data where the practice of using classical distributions for the purpose was found to be inappropriate. The weighted distributions arise when observations generated from a stochastic process are not given equal chances of being recorded hence they are recorded according to some weight function. The weighted distributions provide an integrative approach to deal with model specification and data interpretation problems. The weighted distribution gives a method for fitting models to the unknown weight function when the samples can be taken both from original and developed distributions. The weighted distributions have been employed in many fields of real life such as biomedicine, ecology, survival analysis, reliability, analysis of family data, forestry and other areas for the development of proper statistical models. The concept of weighted probability models attracted a lot of researchers to contemplate on and to carry out research on this topic. The weighted distributions will occur in a natural way in specifying probabilities of events as observed and recorded by making adjustment to probabilities of actual occurrence of events taking into account the method of ascertainment. Failure to make such adjustments can lead to wrong conclusion.

There are various sources which provide detailed description of weighted distributions. Patil and Rao (1978) studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Warren (1975) was first to apply the size biased distributions in connection with sampling wood cells. Eyob et al. (2019) presented weighted quasi Akash distribution with properties and applications. Manoj and Elangovan (2019) presented weighted Odoma distribution with properties and applications in cancer data. Mudasir et al. (2018) studies the weighted new weibull Pareto distribution with characterizations and applications. Mudasir and Ahmad (2017) introduced the weighted erlang distribution with characterization and information measures. Atikankul (2023) presented Bayesian inference for a weighted Bilal distribution with regression model. Ahmad and Ahmad (2019) discussed on weighted analogue of inverse gamma distribution with statistical properties, estimation and simulation study. Modi and Gill (2015) introduced length biased weighted Maxwell distribution. Iqbal and Zafar Iqbal (2020) introduced the mixture of weighted exponential and weighted gamma distribution. Dar et al. (2022) proposed poisson weighted Pranav distribution applicable to count data. Jan, Fatima and Ahmad (2017) discussed on weighted Ailamujia distribution with applications to lifetime data. Recently, Ganaie et al. (2023) studied the weighted power Garima distribution with applications in blood cancer and relief times.

Juchez distribution is a newly proposed one parametric probability distribution introduced by Echebiri and Mbegbu (2022). Its different statistical properties like moments, coefficient of variation, skewness, kurtosis, index of dispersion, mean residual life function, hazard function, moment generating function, stochastic ordering, order statistics, Renyi entropy, bonferroni and lorenz curves have been studied. Further, its parameters have been estimated by using the method of maximum likelihood estimation.

2. Weighted Juchez (WJ) Distribution

The probability density function of Juchez distribution is given by

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+6}\left(1+x+x^3\right)e^{-\theta x};\ x>0,\ \theta>0 \qquad (1)$$

and the cumulative distribution function of Juchez distribution is given by

$$F(x;\theta)=1-\left(1+\frac{\theta x[\theta^2+\theta^2x^2+3\theta x+6]}{\theta^3+\theta^2+6}\right)e^{-\theta x};\quad x>0,\ \theta>0 \qquad (2)$$

Suppose X be the non-negative random variable with probability density function $f(x)$ .Let $w(x)$ be its non-negative weight function, then the probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}, \quad x > 0.$$

Where $w(x)$ be the non - negative weight function and $E(w(x)) = \int w(x) f(x) dx < \infty$.

For various forms of weight function $w(x)$ especially when $w(x) = x^c$, resulting distribution is termed as weighted distribution. In this paper, we have to obtain the weighted version of Juchez distribution, so the weight function at $w(x) = x^c$, resulting distribution is known as weighted Juchez distribution and its probability density function is given by

$$f_w(x) = \frac{x^c f(x)}{E(x^c)} \tag{3}$$

Where $E(x^c) = \int_0^\infty x^c f(x) dx$

$$E(x^c) = \frac{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)}{\theta^c (\theta^3 + \theta^2 + 6)} \tag{4}$$

Substituting the equations (1) and (4) in equation (3), we will obtain the probability density function of weighted Juchez distribution as

$$f_w(x) = \frac{x^c \theta^{c+4}}{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)} \left(1 + x + x^3\right) e^{-\theta x} \tag{5}$$

and the cumulative distribution function of weighted Juchez distribution can be obtained as

$$F_w(x) = \int_0^x f_w(x) dx$$

$$F_w(x) = \int_0^x \frac{x^c \theta^{c+4}}{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)} \left(1 + x + x^3\right) e^{-\theta x} dx$$

$$F_w(x) = \frac{1}{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)} \int_0^x x^c \theta^{c+4} \left(1 + x + x^3\right) e^{-\theta x} dx$$

$$F_w(x) = \frac{1}{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)} \left(\begin{array}{l} \theta^{c+4} \int_0^x x^c e^{-\theta x} dx + \theta^{c+4} \int_0^x x^{c+1} e^{-\theta x} dx \\ \\ + \theta^{c+4} \int_0^x x^{c+3} e^{-\theta x} dx \end{array} \right) \tag{6}$$

Put $\theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$, Also $x = \frac{t}{\theta}$

When $x \to x,\ t \to \theta x$ and When $x \to 0,\ t \to 0$

After the simplification of equation (6), we will obtain the cumulative distribution function of weighted Juchez distribution as

$$F_w(x) = \frac{1}{\theta^3 \Gamma(c+1) + \theta^2 \Gamma(c+2) + \Gamma(c+4)} \left(\theta^3 \gamma(c+1, \theta x) + \theta^2 \gamma(c+2, \theta x) + \gamma(c+4, \theta x)\right) \tag{7}$$

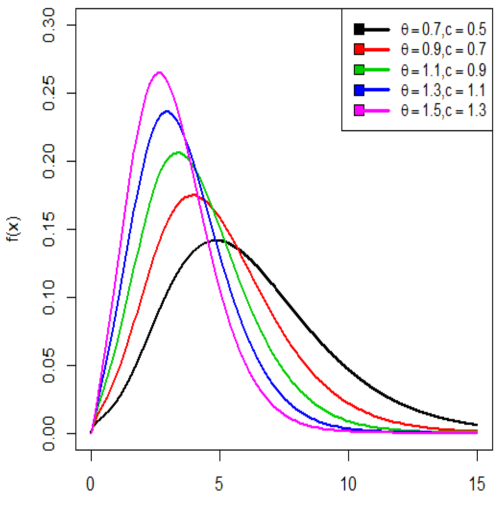

Fig.1:Pdf plot of Weighted Juchez Distribution

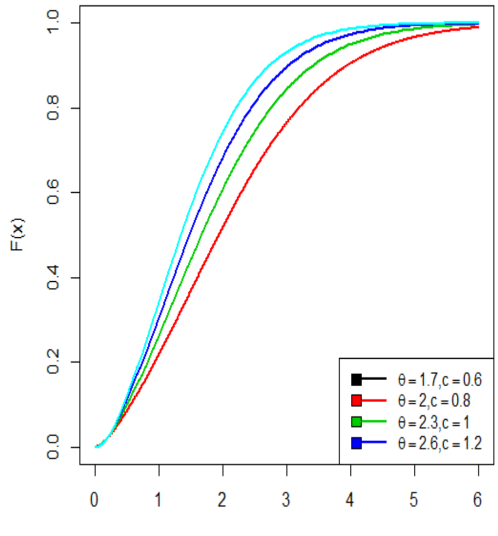

Fig.2:cdf plot of Weighted Juchez Distribution

3. Survival Analysis

In this section, we will discuss about the survival function, hazard rate function and reverse hazard rate function of proposed weighted Juchez distribution.

3.1 Survival function

The survival function of weighted Juchez distribution can be obtained as

$$S(x) = 1 - F_w(x)$$

$$S(x) = 1 - \frac{1}{\theta^3\Gamma(c+1) + \theta^2\Gamma(c+2) + \Gamma(c+4)}\left(\theta^3\gamma(c+1,\, \theta x) + \ \theta^2\gamma(c+2,\, \theta x) + \gamma(c+4,\, \theta x)\right)$$

3.2 Hazard function

The hazard function is also known as failure rate or force of mortality and is given by

$$h(x) = \frac{f_w(x)}{S(x)}$$

$$h(x) = \frac{x^c\theta^{c+4}(1+x+x^3)\, e^{-\theta x}}{(\theta^3\Gamma(c+1) + \theta^2\Gamma(c+2) + \Gamma(c+4)) - (\theta^3\gamma(c+1,\, \theta x) + \theta^2\gamma(c+2,\, \theta x) + \gamma(c+4,\, \theta x))}$$

3.3 Reverse hazard function

The reverse hazard rate function is given by

$$h_r(x) = \frac{f_w(x)}{F_w(x)}$$

$$h_r(x) = \frac{x^c\theta^{c+4}(1+x+x^3)\, e^{-\theta x}}{(\theta^3\gamma(c+1,\, \theta x) + \theta^2\gamma(c+2,\, \theta x) + \gamma(c+4,\, \theta x))}$$

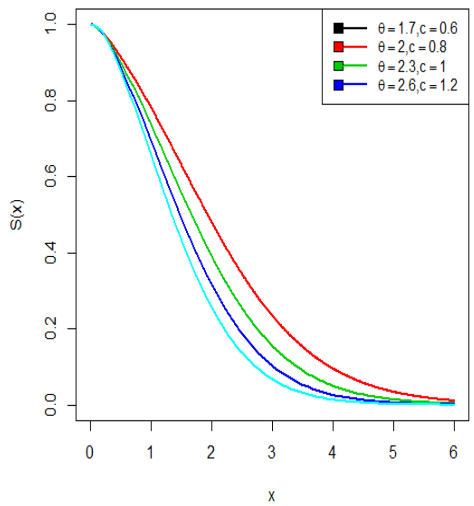

Fig.3:Survival plot of Weighted Juchez Distribution

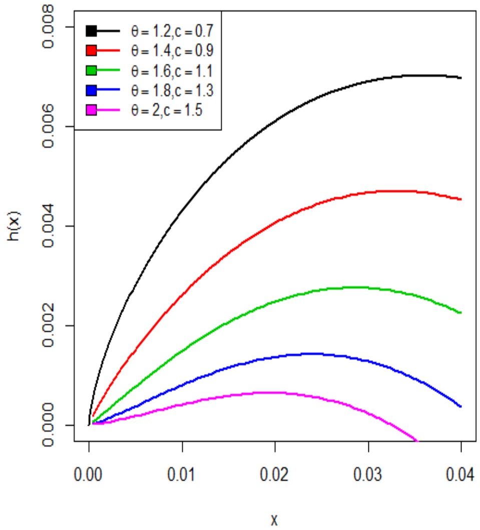

Fig.4:Hazard Plot of Weighted Juchez Distribution

4. Statistical Properties

In this section, we will discuss the different statistical properties of weighted Juchez distribution that include moments, harmonic mean, moment generating function and characteristic function.

4.1 Moments

Let X denotes the random variable represents weighted Juchez distribution with parameters θ and c, then the r^{th} order moment $E(X^r)$ of proposed weighted Juchez distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_w(x)dx$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r \frac{x^c \theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \int_0^\infty \frac{x^{c+r} \theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \frac{\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\int_0^\infty x^{c+r}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \frac{\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\begin{array}{l}\int_0^\infty x^{(c+r+1)-1}e^{-\theta x}dx + \int_0^\infty x^{(c+r+2)-1}e^{-\theta x}dx \\ + \int_0^\infty x^{(c+r+4)-1}e^{-\theta x}dx\end{array}\right) \tag{8}$$

After the simplification of equation (8), we obtain

$$E(X^r) = \mu_r' = \frac{\theta^3\Gamma(c+r+1)+\theta^2\Gamma(c+r+2)+\Gamma(c+r+4)}{\theta^r(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))} \tag{9}$$

By substituting r = 1, 2, 3 and 4 in equation (9), we will obtain the first four moments of weighted Juchez distribution as

$$E(X) = \mu_1' = \frac{\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5)}{\theta(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}$$

$$E(X^2) = \mu_2' = \frac{\theta^3\Gamma(c+3)+\theta^2\Gamma(c+4)+\Gamma(c+6)}{\theta^2(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}$$

$$E(X^3)=\mu_3'=\frac{\theta^3\Gamma(c+4)+\theta^2\Gamma(c+5)+\Gamma(c+7)}{\theta^3(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}$$

$$E(X^4)=\mu_4'=\frac{\theta^3\Gamma(c+5)+\theta^2\Gamma(c+6)+\Gamma(c+8)}{\theta^4(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}$$

$$\text{Variance }=\frac{\theta^3\Gamma(c+3)+\theta^2\Gamma(c+4)+\Gamma(c+6)}{\theta^2(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}-\left(\frac{\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5)}{\theta(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}\right)^2$$

$$S.D(\sigma)=\sqrt{\left(\frac{\theta^3\Gamma(c+3)+\theta^2\Gamma(c+4)+\Gamma(c+6)}{\theta^2(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}-\left(\frac{\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5)}{\theta(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}\right)^2\right)}$$

4.2 Harmonic mean

The harmonic mean for proposed weighted Juchez distribution can be obtained as

$$H.M=E\left(\frac{1}{x}\right)=\int_0^\infty\frac{1}{x}f_w(x)dx$$

$$H.M=\int_0^\infty\frac{1}{x}\frac{x^c\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$H.M=\int_0^\infty\frac{x^{c-1}\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$H.M=\frac{\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\int_0^\infty x^{c-1}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$H.M=\frac{\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\int_0^\infty x^{(c+1)-2}e^{-\theta x}dx+\int_0^\infty x^{(c+1)-1}e^{-\theta x}dx+\int_0^\infty x^{(c+3)-1}e^{-\theta x}dx\right)$$

After the simplification of above equation, we obtain

$$H.M=\frac{\theta(\theta^2\Gamma(c+1)+\theta^2\Gamma(c+1)+\Gamma(c+3))}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}$$

4.3 Moment generating function and characteristic function

Suppose the random variable X follows weighted Juchez distribution with parameters θ and c, then the moment generating function of executed distribution can be obtained as

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f_w(x)dx$$

Using Taylor's series, we obtain

$$M_X(t)=\int_0^\infty\left(1+tx+\frac{(tx)^2}{2!}+\ldots\right)f_w(x)dx$$

$$M_X(t)=\int_0^\infty\sum_{j=0}^\infty\frac{t^j}{j!}x^jf_w(x)dx$$

$$M_X(t)=\sum_{j=0}^\infty\frac{t^j}{j!}\mu_j'$$

$$M_X(t)=\sum_{j=0}^\infty\frac{t^j}{j!}\left(\frac{\theta^3\Gamma(c+j+1)+\theta^2\Gamma(c+j+2)+\Gamma(c+j+4)}{\theta^j(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}\right)$$

$$M_X(t)=\frac{1}{(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}\sum_{j=0}^{\infty}\frac{t^j}{j!\theta^j}\left(\theta^3\Gamma(c+j+1)+\theta^2\Gamma(c+j+2)+\Gamma(c+j+4)\right)$$

Similarly characteristic function of weighted Juchez distribution can be obtained as

$$\varphi_x(t)=M_X(it)$$

$$M_X(it)=\frac{1}{(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}\sum_{j=0}^{\infty}\frac{it^j}{j!\theta^j}\left(\theta^3\Gamma(c+j+1)+\theta^2\Gamma(c+j+2)+\Gamma(c+j+4)\right)$$

5. Order Statistics

Suppose $X_{(1)}, X_{(2)},\ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2,\ldots, X_n$ from a continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_X(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r} \tag{10}$$

By using the equations (5) and (7) in equation (10), we will obtain the probability density function of r^{th} order statistics of weighted Juchez distribution as

$$f_{x(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\left(\frac{x^c\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}\right)$$

$$\times\left(\frac{1}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\theta^3\gamma(c+1,\theta x)+\theta^2\gamma(c+2,\theta x)+\gamma(c+4,\theta x)\right)\right)^{r-1}$$

$$\times\left(1-\frac{1}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\theta^3\gamma(c+1,\theta x)+\theta^2\gamma(c+2,\theta x)+\gamma(c+4,\theta x)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of weighted Juchez distribution can be obtained as

$$f_{x(n)}(x)=\frac{n x^c\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}$$

$$\times\left(\frac{1}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\theta^3\gamma(c+1,\theta x)+\theta^2\gamma(c+2,\theta x)+\gamma(c+4,\theta x)\right)\right)^{n-1}$$

and probability density function of first order statistic $X_{(1)}$ of weighted Juchez distribution can be obtained as

$$f_{x(1)}(x)=\frac{n x^c\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x+x^3\right)e^{-\theta x}$$

$$\times\left(1-\frac{1}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(\theta^3\gamma(c+1,\theta x)+\theta^2\gamma(c+2,\theta x)+\gamma(c+4,\theta x)\right)\right)^{n-1}$$

6. Likelihood Ratio Test

Suppose $X_1, X_2, \ldots, X_n$ be the random sample of size n from weighted Juchez distribution. To analyze its significance the hypothesis is to be examined.

$$H_o : f(x) = f(x;\theta) \quad \text{against} \quad H_1 : f(x) = f_w(x;\theta,c)$$

In order to determine, whether random sample of size n comes from Juchez distribution or weighted Juchez distribution, the following statistic rule is used.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_w(x;\theta,c)}{f(x;\theta)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \left(\frac{x_i^{\,c}\theta^c(\theta^3+\theta^2+6)}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)} \right)$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta^c(\theta^3+\theta^2+6)}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)} \right)^n \prod_{i=1}^{n} x_i^{\,c}$$

We should refuse to accept the null hypothesis, if

$$\Delta = \left(\frac{\theta^c(\theta^3+\theta^2+6)}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)} \right)^n \prod_{i=1}^{n} x_i^{\,c} > k$$

Equivalently we should also reject the null hypothesis, where

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k \left(\frac{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}{\theta^c(\theta^3+\theta^2+6)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^*, \text{Where } k^* = k \left(\frac{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}{\theta^c(\theta^3+\theta^2+6)} \right)^n$$

Thus for the large sample of size n, $2\log \Delta$ is distributed as chi-square distribution with one degree of freedom and also chi-square distribution is applied for determining the p-value. Thus, we should refuse to accept the null hypothesis if the probability value is given by

$p\left(\Delta^* > \alpha^*\right)$, Where $\alpha^* = \prod_{i=1}^{n} x_i^{\,c}$ is smaller than a specified level of significance and $\prod_{i=1}^{n} x_i^{\,c}$ is the observed value of the statistic Δ^*.

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are oldest classical curves are used in economics to measure the distribution of inequality in income or poverty. The bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f_w(x)dx$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f_w(x)dx \quad \text{and} \quad q = F^{-1}(p)$$

$$\text{Where } \mu_1' = \frac{\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5)}{\theta(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}$$

$$B(p) = \frac{\theta(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))}{p(\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5))} \int_0^q \frac{x^{c+1}\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)} \left(1+x+x^3\right)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^{c+5}}{p(\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5))}\int_0^q x^{c+1}\left(1+x+x^3\right)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^{c+5}}{p(\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5))}\left(\int_0^q x^{(c+2)-1}e^{-\theta x}dx+\int_0^q x^{(c+3)-1}e^{-\theta x}dx+\int_0^q x^{(c+5)-1}e^{-\theta x}dx\right)$$

After the simplification of above equation, we obtain

$$B(p)=\frac{\theta^{c+5}}{p(\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5))}\left(\gamma(c+2,\,\theta q)+\gamma(c+3,\,\theta q)+\gamma(c+5,\,\theta q)\right)$$

$$L(p)=\frac{\theta^{c+5}}{(\theta^3\Gamma(c+2)+\theta^2\Gamma(c+3)+\Gamma(c+5))}\left(\gamma(c+2,\,\theta q)+\gamma(c+3,\,\theta q)+\gamma(c+5,\,\theta q)\right)$$

8. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the parameter estimation of weighted Juchez distribution by employing the technique of maximum likelihood estimation. Consider $X_1, X_2, \ldots, X_n$ be a random sample of size n from the weighted Juchez distribution, then the likelihood function can be written as

$$L(x)=\prod_{i=1}^{n} f_w(x)$$

$$L(x)=\prod_{i=1}^{n}\left(\frac{x_i^{\,c}\,\theta^{c+4}}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\left(1+x_i+x_i^{\,3}\right)e^{-\theta x_i}\right)$$

$$L(x)=\frac{\theta^{n(c+4)}}{(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))^n}\prod_{i=1}^{n}\left(x_i^{\,c}\left(1+x_i+x_i^{\,3}\right)e^{-\theta x_i}\right)$$

The log likelihood function is given by

$$\log L=n(c+4)\log\theta-n\log(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))+c\sum_{i=1}^{n}\log x_i$$

$$+\sum_{i=1}^{n}\log\left(1+x_i+x_i^{\,3}\right)-\theta\sum_{i=1}^{n}x_i \qquad (11)$$

Now differentiating the log likelihood equation (11) with respect to parameter θ and c. we must satisfy the following normal

equations

$$\frac{\partial\log L}{\partial\theta}=\frac{n(c+4)}{\theta}-n\left(\frac{3\theta^2\Gamma(c+1)+2\theta\Gamma(c+2)}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\right)-\sum_{i=1}^{n}x_i=0$$

$$\frac{\partial\log L}{\partial c}=n\log\theta-n\psi\left(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)\right)+\sum_{i=1}^{n}\log x_i=0$$

Where ψ (.) is the digamma function.

Because of complicated form of above likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore, we use R and wolfram mathematics for estimating the parameters of proposed distribution.

In order to obtain confidence interval, we use the asymptotic normality results. We have that if $\hat{\alpha}=(\hat{\theta},\,\hat{c})$ denotes the MLE *of* $\alpha=(\theta,\,c)$. We can analyze the results as

$$\sqrt{n}(\hat{\alpha}-\alpha)\to N_2(0,\,I^{-1}(\alpha))$$

Where $I^{-1}(\alpha)$ is Fisher's information matrix.i.e.,

$$I(\alpha) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

Here, we define

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n(c+4)}{\theta^2} - n\left(\frac{(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))\,(6\theta\Gamma(c+1)+2\Gamma(c+2)) - (3\theta^2\Gamma(c+1)+2\theta\Gamma(c+2))(3\theta^2\Gamma(c+1)+2\theta\Gamma(c+2))}{(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4))^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'\left(\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n}{\theta} - n\psi\left(\frac{3\theta^2\Gamma(c+1)+2\theta\Gamma(c+2)}{\theta^3\Gamma(c+1)+\theta^2\Gamma(c+2)+\Gamma(c+4)}\right)$$

Since α being unknown, we estimate $I^{-1}(\alpha)$ by $I^{-1}(\hat{\alpha})$ and this can be used to obtain asymptotic confidece interval for θ and c.

9.Application

In this section, we have fitted a real lifetime data set in weighted Juchez distribution in order to show that the weighted Juchez distribution fits better over Juchez, exponential and Lindley distributions. The real lifetime data set is given below as.

The following real lifetime data set represents the remission time (in months) of 50 breast cancer women subjected to treatment by using trastzuzumab as medication reported by cancer registry department, university of Benin teaching hospital, Benin, Edo state and the real lifetime data set is given as under in table 1.

Table 1: Data regarding the remission time (in months) of 50 breast cancer women represented by cancer registry department

50	74	35	39	21	37	27	35	30	35
26	38	34	34	26	41	61	33	33	26
25	41	35	34	34	33	60	61	42	30
80	31	24	49	26	31	28	41	37	41
61	33	26	34	50	73	45	80	39	21

To estimate the unknown parameters along with the model comparison criterion values, the technique of R software is used. In order to compare the performance of weighted Juchez distribution over Juchez, exponential and Lindley distributions, we use the criterion values like *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected) and *-2logL*. The better distribution is which corresponds to lesser values of *AIC*, *BIC*, *AICC* and *-2logL*. For computing the criterion values *AIC*, *BIC*, *AICC* and *-2logL*, following formulas are used.

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \qquad \text{and} \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Where k is number of parameters in statistical model, n is the sample size and $-2\log L$ is the maximized value of log-likelihood function under the considered model.

Table 2: Shows Performance of fitted distributions

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Weighted Juchez	$\hat{\theta} = 0.2189351$ $\hat{c} = 4.6717031$	$\hat{\theta} = 0.0441932$ $\hat{c} = 1.6996744$	397.8634	401.8634	405.6875	402.1187
Juchez	$\hat{\theta} = 0.10091489$	$\hat{\theta} = 0.00713114$	410.2831	412.2831	414.1952	412.3664
Exponential	$\hat{\theta} = 0.02526572\ 2$	$\hat{\theta} = 0.00356750\ 5$	467.8829	469.8829	471.7949	469.9662
Lindley	$\hat{\theta} = 0.04931800\ 7$	$\hat{\theta} = 0.00493249\ 5$	437.2288	439.2288	441.1409	439.3121

From results given above in table 2, it has been clearly seen that the weighted Juchez distribution has lesser *AIC*, *BIC*, *AICC* and -*2logL* values as compared to the Juchez, exponential and Lindley distributions. Hence, it can be realized that the weighted Juchez distribution leads to a better fit over Juchez, exponential and Lindley distributions.

10. Conclusion

The present manuscript deals with a new probability model of Juchez distribution called as weighted Juchez distribution. The subject distribution is introduced by using the weighted technique to its baseline distribution. The proposed new distribution is discussed with its different structural properties that include moments, the mean and variance, survival function, hazard function, moment generating function, reverse hazard function, order statistics, bonferroni and lorenz curves. Its parameters have also been estimated by using the technique of maximum likelihood estimation. Finally, a new distribution has been analysed and investigated with real lifetime data set to examine its superiority and hence it is found from the result that proposed weighted Juchez distribution leads to a better fit than the Juchez, exponential and Lindley distributions.

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