New Higher Order Derivative Runge-Kutta Methods for Solving Ordinary Differential Equations

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Abstract— **The derivation of numerical methods to deal with real life problems framed into differential equations has been on the increase of which good attention is directed to Runge-Kutta methods. Recently, researchers have explored the derivation of Runge-Kutta methods by introducing higher order derivative $\mathbf{f'}$ (up to the second order) in the Runge-Kutta $\mathbf{k_i}$ terms $(\mathbf{i > 1})$ in order to increase the order of accuracy of the solution to the differential equation. However, this paper presents some new higher order derivative Runge-Kutta methods by using other types of 'mean' such as harmonic mean, geometric mean or harmonic mean as against the conventional Arithmetic mean viewed higher order derivative Runge-Kutta methods. The qualitative features such as the local truncation error, consistency, convergence and stability of the new methods are investigated and established. Numerical examples are used to compare the accuracies of these methods. The results show better performance of some of the derived methods when compared with existing methods.**

Index Terms—**Multiderivative, Harmonic Mean, Geometric Mean, Heronian Mean, Differential Equations**

I. Introduction

In biological sciences, physical sciences and engineering, mathematical models are composed to assist in the interpretation of physical phenomena. These composed models oftentimes result to equations which contain the derivatives of an unknown function. In applications, the function usually represent physical quantities and the derivatives represent their rates of change. These equations are called differential equations. Ordinary differential Equations $(ODEs)$ arise on many occasions when using mathematical modelling techniques to describe these physical phenomena. The general form of an initial value problems $(IVPs)$ in ODE is of the form:

$$y' = f(x,y), \qquad y(x_0) = \eta \qquad (1)$$

where $x \in \mathbb{R}$, $y, \eta \in \mathbb{R}^n$ and $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The development of numerical methods for the solution of $ODEs$ have turned out to be a very rapid research area in recent decades because of the difficulties encountered in finding analytical solutions to some mathematical models composed into differential equations from real life situations of which good attention is directed to Runge-Kutta methods.

In recent times, so much work have been done by researchers to improve the Runge Kutta methods for solving O.D.E. Several methods have been developed using the idea of different types of mean such as the geometric mean, heronian mean, centroidal mean, contra-harmonic mean and harmonic mean. Akanbi M.A. [4] proposed a 3-stage geometric explicit Runge-Kutta methods for singular autonomous initial value problems in ordinary differential equations where geometric mean was incorporated in the classical 3-stage Runge-Kutta methods. A third order harmonic mean for autonomous initial value problem was constructed by Wusu A.S., Okunuga S.A. and Shofoluwe A.B. [18]. The method was derived based on harmonic mean and was confirmed to be better than any third order of any form of explicit Runge-Kutta methods. This idea was extended to fourth order in Wusu A.S., Akanbi M.A. and Bakre F.O. [17]. Olaniyan A.S. *et al.* [15] constructed a new Implicit Runge-Kutta method in which heronian mean was used as a basis in the derivation. The paper was found to perform better than the classical 2-Stage Implicit Runge-Kutta methods.

In an earlier research work of Goeken D. and Johnson O.[10], a 2-stage explicit Runge-Kutta method of order 3 was developed for autonomous Initial Value Problems with the notion of incorporating first derivative in the internal stages of Runge-Kutta method. This method was later extended to fourth and fifth order methods in Goeken D. and Johnson O. [11]. Akanbi [3] improved on this research by deriving multi-derivative explicit Runge-Kutta method involving first and second derivatives which provided better results. Wusu *et al.* [19] then present a new class of three stage Runge-Kutta method with first and second derivatives of which the cost of internal stage evaluations is reduced greatly and there is an appreciable improvement on the attainable order of accuracy of the method. Several authors such as Vijeyata C. and Pankaj K. [16], Mukaddes O. T. and Turgut O. [13], Bazuaye, F. E. [6], Chan, R. P. and Tsai, A. Y. [7], Aiguobasimwin, I.B. and Okuonghae, R.I. [2] and Adeyeye, O. *et al.* [1] to mention a few have developed methods based on higher derivatives Runge-Kutta methods and Runge-Kutta methods with the notion of other types of 'mean'. However, this research is motivated by the need to find a common ground that will harness the strength of these methods to produce new methods. By following closely these techniques, some new 3 - stage explicit Runge-Kutta methods are constructed based on different types of 'mean' such as harmonic mean, geometric mean or heronian mean and higher derivatives up to the second derivative in the k_i terms of Runge-Kutta method on a single explicit Runge-Kutta methods which have previously been done on different explicit Runge-Kutta methods in order to achieve a higher order of accuracy on a single Runge-Kutta method that would be an improvement on the existing methods; be potent enough to compete favorably in the solution of IVP of O.D.E.; be less expensive in terms of the number of functions evaluation per step; be consistent, convergent and stable.

II. DERIVATION OF THE METHODS

In this research, a combination of other types of 'mean' such as geometric mean, harmonic mean or heronian mean with higher derivatives up to the second derivative in a single Runge-Kutta Methods are derived. The methods can be expressed in a general form as follows:

$$
\begin{aligned}
y_{n+1} - y_n &= \Phi(y_n; h) \quad (2)\\
k_1 &= hf(y)\\
k_2 &= hf(y + hb_{21}k_1 + h^2 b_{22} f f_y + \tfrac{h^3}{2} b_{23}(f f_y^2 + f^2 f_{yy}))\\
k_3 &= hf(y + hb_{31}k_1 + hb_{32}k_2 + h^2 b_{33} f f_y + \tfrac{h^2}{2} b_{34}(f f_y^2 + f^2 f_{yy}))
\end{aligned}
$$

Where $\Phi(y_n; h)$ will be equivalent to:

$$\Phi_{GMERK}(y_n; h) = c_1\sqrt{k_1 k_2} + c_2\sqrt{k_2 k_3} \quad , \tag{3}$$

$$\Phi_{HaMERK}(y_n; h) = c_1 \frac{2k_1 k_2}{k_1 + k_2} + c_2 \frac{2k_2 k_3}{k_2 + k_3} \tag{4}$$

and

$$\Phi_{HeMERK}(y_n; h) = c_1 \frac{k_1 + 2k_2 + k_3 + \sqrt{k_1 k_2} + \sqrt{k_2 k_3}}{6} \tag{5}$$

To derive these schemes, the following steps will be adopted:

- Obtain the Taylor's series expansion of k_2 and k_3.
- Insert k_1 and the Taylor's series expansions of k_2 and k_3 into $\Phi_{GMERK}(y_n; h)$, $\Phi_{HAMERK}(y_n; h)$and $\Phi_{HEMERK}(y_n; h)$.
- Compare the results with Taylor's series expansion of y_{n+1} about (x_n, y_n) up to order $O(h^4)$ to obtain three different systems of equations which were solved to obtain corresponding parameters.

These parameters are then substituted into the general forms to obtain the following methods:

- Geometric-Multiderivative Explicit Runge-Kutta Method (GMERK)
-

$$y_{n+1} - y_n = \frac{2}{9}\sqrt{k_1 k_2} + \frac{7}{9}\sqrt{k_2 k_3}$$

where

$$
\begin{aligned}
k_1 &= hf(y)\\
k_2 &= hf\left(y + hk_1 - \frac{5}{8}(f f_y^2 + f^2 f_{yy})\right)\\
k_3 &= hf\left(y + \frac{1}{3}hk_1 - \frac{2}{3}h^2 f f_y + \frac{3}{10}h^3(f f_y^2 + f^2 f_{yy})\right)
\end{aligned}
$$

- Harmonic-Multiderivative Explicit Runge-Kutta Method (HaMERK)
-

$$y_{n+1} - y_n = \frac{k_1 k_2}{3(k_1 + k_2)} + \frac{2k_2 k_3}{3(k_2 + k_3)}$$

where

$$
\begin{aligned}
k_1 &= hf(y)\\
k_2 &= hf\left(y + hk_1 + \frac{1}{2}h^2 f f_y + \frac{5}{2}(f f_y^2 + f^2 f_{yy})\right)\\
k_3 &= hf\left(y + \frac{1}{3}hk_1 - \frac{2}{3}h^2 f f_y + \frac{8}{3}h^3(f f_y^2 + f^2 f_{yy})\right)
\end{aligned}
$$

- Heronian-Multiderivative Explicit Runge-Kutta Method (HeMERK)
-

$$y_{n+1} - y_n = \frac{k_1 + 2k_2 + k_3 + \sqrt{k_1 k_2} + \sqrt{k_2 k_3}}{6}$$

where

$$
\begin{aligned}
k_1 &= hf(y)\\
k_2 &= hf\left(y + hk_1 + \frac{2}{5}h^2 f f_y + \frac{3}{10}(f f_y^2 + f^2 f_{yy})\right)\\
k_3 &= hf\left(y + \frac{5}{6}hk_1 - \frac{5}{3}h^2 f f_y + \frac{11}{6}h^3(f f_y^2 + f^2 f_{yy})\right)
\end{aligned}
$$

III. QUALITATIVE FEATURES

We will consider some basic features which are very vital to the development of the constructed schemes. These features are local truncation error, consistency, stability and convergence.

3.1 Local Truncation Error

Definition 3.1 (Lambert [12])

The *local truncation error* T_{n+1} at x_{n+1} of the general one step method is given as

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n), h)$$

where $y(x_n)$ is the theoretical solution.

The local truncation error of the constructed schemes in compliance with the above definition can be expressed as

$$T_{n+1} = y(x_{n+1}) - y_{n+1}$$

Definition 3.2 (Lambert [12])

A numerical method is said to be of order p if p is the largest integer for which $T_{n+1} = O(h^{p+1})$ for every n and $p \geq 1$.

Consequently, the local truncation error of the methods constructed in this research work are as follows:

- *3-Stage GMERK*

$$T_{n+1} = \frac{h^5}{480}\left(2ff_y^4 - 5f^3f_{yy}^2 + f^2f_y^2f_{yy} - 3f^3f_yf_{yyy} - 4f^4f_{yyyy}\right)$$

- *3-Stage HaMERK*

$$T_{n+1} = \frac{h^5}{720}\left(-6ff_y^4 - 5f^3f_{yy}^2 + 15f^2f_y^2f_{yy} - 7f^3f_yf_{yyy} - 2f^4f_{yyyy}\right)$$

- *3-Stage HeMERK*

$$T_{n+1} = \frac{h^5}{1920}\left(-3ff_y^4 - 5f^3f_{yy}^2 + 11f^2f_y^2f_{yy} - 7f^3f_yf_{yyy} - 3f^4f_{yyyy}\right)$$

Theorem 3.3 (Lambert J. D. (1991))

Let $f(x, y)$ belongs to $C^3[a, b]$ and let its partial derivatives be bounded and if $\exists\ L, M$ some positive constants such that

$$|f(x, y)| < M, \qquad \left|\frac{\delta^{i+j}}{\delta x^i \delta y^j}\right| < \frac{L^{i+j}}{M^{i-j}}, \qquad i + j < M$$

then in terms of error bound by virtue of Lotkin in Lambert [12],

hence the strict upper bound with respect to y for the derived methods are:

$$|LTE_{Gm}| \leq \frac{9}{480}h^4ML^3 + O(h)^5$$
$$|LTE_{Ha}| \leq \frac{5}{720}h^4ML^3 + O(h)^5$$
$$|LTE_{He}| \leq \frac{7}{1920}h^4ML^3 + O(h)^5$$

3.2 Consistency

Definition 3.4 (Lambert [12])

A numerical method is said to be consistent with an initial value problem if

$$\phi(x, y, 0) \equiv f(x, y)$$

Thus, a consistent method has at least order one.

Definition 3.5 (Lambert [12])

A scheme is said to be consistent if the difference equation of the integrating formula exactly approximates the differential equation it intends to solve as the step size approaches zero.

In order to establish the consistency property of the proposed scheme it is sufficient to show that

$$\lim_{h\to 0}\phi(x_n, y_n; h) = f(x_n, y_n)$$

where $\phi(x_n, y_n; h)$ is the increment function of the numerical scheme.

The consistency of the derived methods were investigated using the above consistency definitions and were all confirmed consistent.

3.3 Stability of the Derived Methods

The stability of numerical methods for solving an IVP in ODE can be analyzed using the linear test problem $y' = \lambda y$ proposed by Dalquist [8], where the solution is $y = e^{\lambda y}$ and λ a complex variable. The stability polynomials $R(z) = \frac{y_{n+1}}{y_n}$ of the derived methods where $z = \lambda h$ are incidentally the same as expressed below:

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

The absolute stability region is as follows:

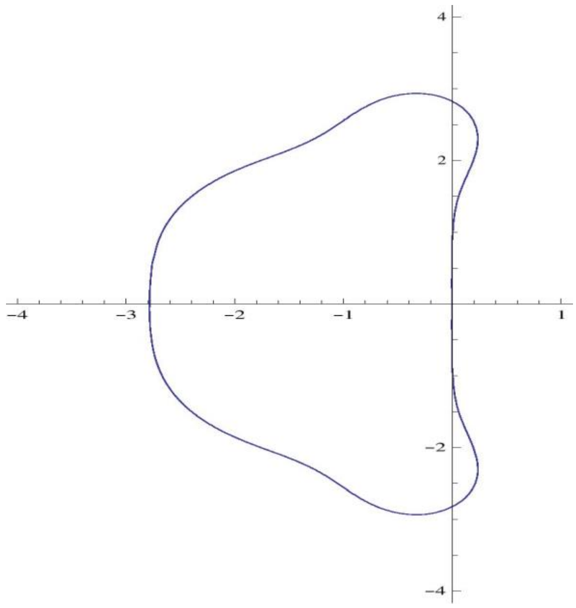

Fig. 1 Absolute Stability of the Methods

3.4 Convergence

We will test for the convergence of the derived methods using the following definitions and theorem.

Definition 3.6 (Dahlquist [8])

A numerical method is said to be convergent if for all initial value problems satisfying the hypothesis of the Lipschitz condition given by

$$|f(x,y) - f(x,y^*)| \leq L|y - y^*|$$

where the Lipschitz constant L is denoted by $L = \max\left|f_y(x,y)\right|$.

Definition 3.7 (Dahlquist [8])

The necessary and sufficient conditions for a numerical method to be convergent is for it to be consistent and stable.

Definition 3.8 (Lambert [12])

A numerical method is said to be convergent if it is consistent and has an order greater than one.

From the theorem and definitions above, we can conclude that the derived methods in the thesis are convergent.

IV. NUMERICAL EXPERIMENT

For the purpose of testing the performance and suitability of the derived schemes, some standard initial value problems are solved with the aid of Matlab package. Comparisons are made with some existing methods such as the kutta's Third Order Method *(ERK)*, Goeken's 3-Stage Method *(3GM)* and 3-Stage Multiderivative Explicit Runge-Kutta Methods *(3MERK)* by Wusu *et al.* (2013) to ascertain the level of accuracy of the derived schemes.

The derived schemes are tested on four initial value problems given below:

4.1 *Problem* 1:

Consider the IVP

$$y' = -y(x) \qquad y(0) = 1$$

whose analytical solution is given as:

$$y(x) = e^{-x}$$

Comparison of the absolute error obtained is presented in the figure below.

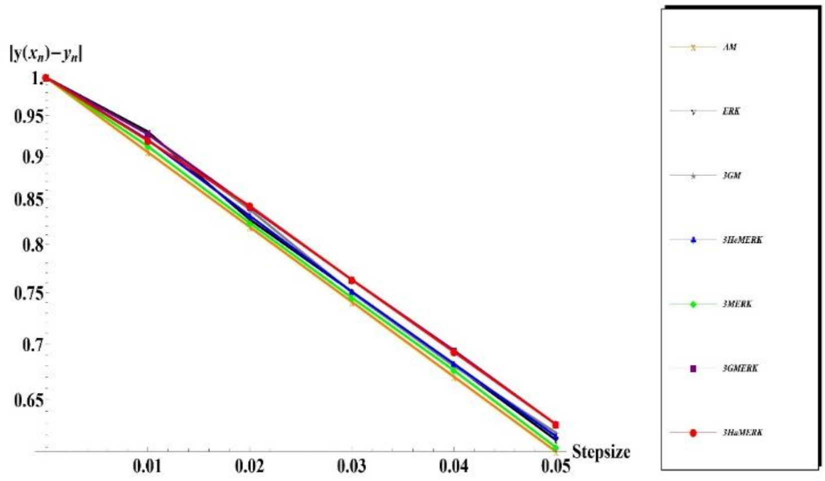

Fig. 2. Absolute Error of Problem 1 using h=0.01

4.2 Problem 2*:*
Consider the IVP

$$y' = x + y(x) \qquad y(0) = 1$$

whose analytical solution is given as:

$$y(x) = 2e^{x} - x - 1$$

Comparison of the absolute error obtained is presented in the figure below.

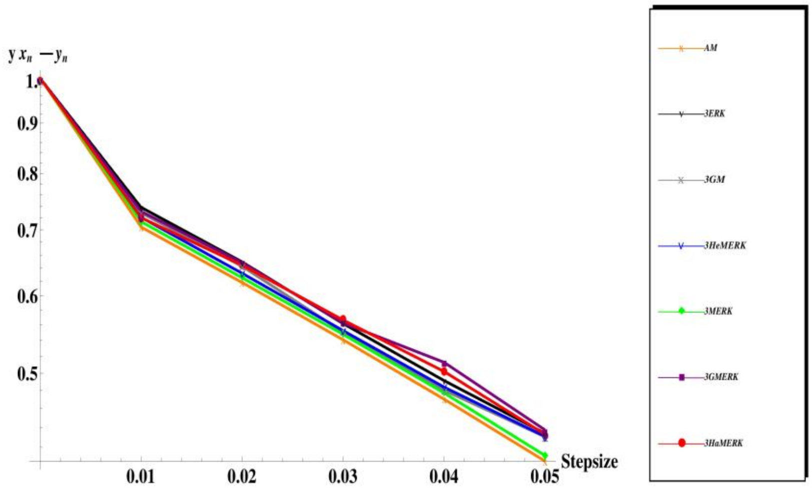

Fig. 3. Absolute Error of Problem 2 using h=0.01

4.3 *Problem* 3*:*
Consider the IVP

$$y' = 1 + \left(y(x)\right)^{2} \qquad y(0) = 2$$

whose analytical solution is given as:

$$y(x) = \tan\left(x + \frac{\pi}{4}\right)$$

Comparison of the absolute error obtained is presented in the figure below.

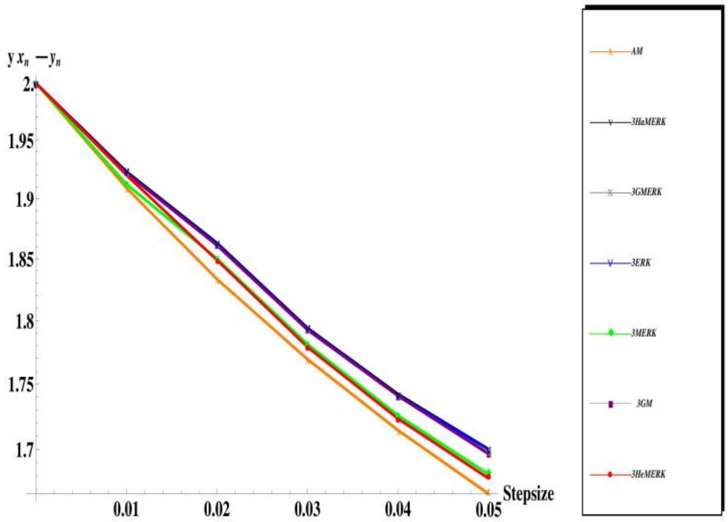

Fig. 4. Absolute Error of Problem 3 using h=0.01

4.4 Problem 4*:*

Consider the IVP

$$y' = -10(y(x) - 1)^2 \qquad y(0) = 2$$

whose analytical solution is given as:

$$y(x) = 1 + \frac{1}{1 + 10x}$$

Comparison of the absolute error obtained is presented in the figure below.

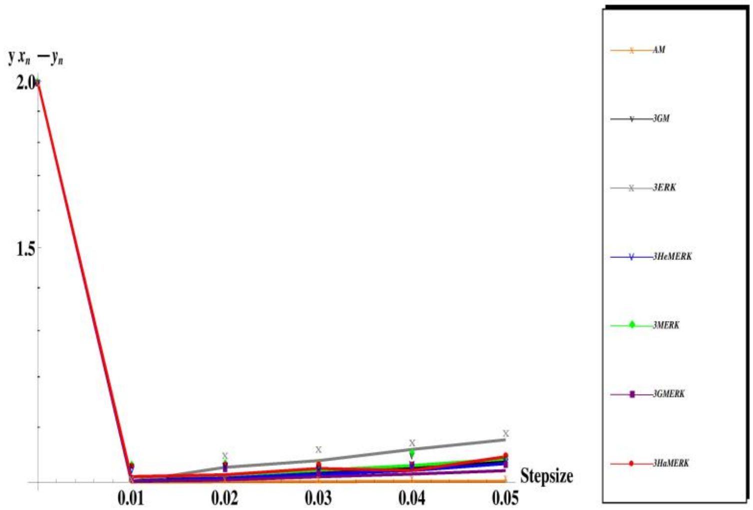

Fig. 5. Absolute Error of Problem 4 using h=0.01

V. Conclusions

VI. In this paper, our research has been devoted to deriving numerical methods to approximate initial value problems in ordinary differential equations. These methods are capable of solving initial value problems arising in various fields of science and engineering. The derivations of the methods are followed by their error analysis: local truncation error, consistency, convergence and stability wherein they exhibited satisfactory performance. The stability regions of the derived methods revealed that the new methods are stable like every other existing numerical methods in the relevant literature. The test problems under consideration were solved by the derived methods and some standard methods of the same stage where it is easy to observe from the figures that some of the new schemes produced smaller errors. Thus, the new derived methods can reliably be used as a substitute for some of the existing 3-stage explicit numerical methods to solve some initial value problems in ordinary differential equations.

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