A Shock Model for Seroconversion Time using Exponentiated Exponential Distribution

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Abstract: In these studies, the results of the seroconversion time of those infected with HIV using the inter arrival times between consecutive contacts are poisson random variables. The purpose of the study is to highlight the time interval between successive contacts have a major role to play in the determination of seroconversion time. In this paper the seroconversion time are dealt with assuming the threshold distribution to be an exponentiated exponential distribution using shock model approach. The statistical measures of seroconversion time are derived and numerical illustrations are provided.

IndexTerms: Antigenic diversity threshold, Seroconversion, Poisson process, Human Immuno Deficiency Virus

I. INTRODUCTION

The outbreak of certain diseases has created that so called epidemic situation in several countries. Human Immuno – deficiency Virus (HIV) infection has created and epidemic situation in several parts of the world simultaneously. HIV that causes AIDS becomes one the world's most series public health challenges. HIV attacks the body's immune system specifically the CD4 cells which helps the immune system off infection. There is no curve for HIV/AIDS but medication can control the infection and prevent disease progression.

The human immune collapses very much based on the antigenic diversity of the antigen namely HIV attained in each successive contacts and it occur leading to seroconversion. Nowak and May (1991) and Stilianakis (1994) have developed mathematical model to predict the antigenic diversity threshold.

Kannan et al (2012, 2017 and 2022) discussed the stochastic model for seroconversion time has been derived different threshold distribution (Mixed Exponential, Exponentiated Modified Weibull, Generalized Rayleigh). A stochastic model for seroconversion time, dealing with the damage process affecting the immune system, which is non-linear and cumulative with the threshold of antigenic diversity, is exponentiated exponential distribution. In this study it is assumed that the inter contact times between successive contact are poisson random variables and the stochastic model is derived for the estimation of seroconversion time. In developing such a stochastic model, cumulative damage process discussed by Esary et al (1973) and a generalized poisson distribution discussed by Anil (2001) are used.

II. Exponentiated Exponential Distribution

Similar to a Weibull or a Gamma family, the Exponentiated Exponential (EE) families have two parameters (scale and shape). Given that many of the characteristics of this new family are relatively similar to those of the Weibull or Gamma families, depending on the shape parameter, this distribution could be utilized as an alternative to a failure rate. The following definition applies to the exponential distribution with exponents.

The distribution function, $F_E(x, \alpha, \lambda)$

$F_E(x, \alpha, \lambda) = (1-e^{-\lambda x})^{\alpha}$; $\alpha, \lambda, x > 0$

Therefore, it has the density function

$f_E(x, \alpha, \lambda) = \alpha \lambda (1-e^{-\lambda x})^{\alpha - 1} e^{-\lambda x}$

The corresponding survival function is

$S_E(x, \alpha, \lambda) = 1 - (1-e^{-\lambda x})^{\alpha}$

Here α is the shape parameter and λ is the scale parameter. When $\alpha = 1$, it represents the exponential family. It can be seen that in a given system if the distribution life time of each component is EE, then the life time of the system itself is EE. While the Weibull distribution is appropriate for a series system, EE represents a parallel system.

III. ASSUMPTIONS OF THE MODEL

1) Sexual contact is the only source of HIV transmission.
2) Every encounter increases the risk of some HIV transmission, which in turn increases antigenic diversity.
3) The process of damage affecting the immune system which is non-linear and cumulative.
4) The time intervals between successive contacts are i.i.d random variables.
5) The damage occur to the system are independent of the threshold level.

IV. RESULTS

The probability density function of Exponentiated Exponential distribution is

$h(y) = \alpha \lambda (1-e^{-\lambda y})^{\alpha - 1} e^{-\lambda y}$ $\alpha, \lambda, y > 0$

and its distribution function is

$H(Y) = (1-e^{-\lambda y})^{\alpha}$ $\alpha, \lambda, y > 0$

Then its survival function $H(\bar{Y}) = 1 - (1-e^{-\lambda y})^{\alpha}$

It can be shown that $P\left[\sum_{i=1}^{k} x_i < y\right] = \int_0^\infty g_k(x)\bar{H}(x)dx$ ------------------ (Eq.1)

Where $\bar{H}(x)$ = 1- H(x)

Put α = 2 then $\bar{H}(y) = 1-(1-e^{-\lambda y})^2$

$\bar{H}(y) = 2\,e^{-\lambda y} - e^{-2\lambda y}$ ------------------ (Eq.2)

Substituting equation (2) in (1), we get

$$P\left[\sum_{i=1}^{k} x_i < y\right] = \int_0^\infty g_k(x)[\,2\,e^{-\lambda x} - e^{-2\lambda x}\,]\,dx$$

$$= [2g^*(\lambda)]^k - [g^*(2\lambda)]^k$$

$$S(t) = P[T>t]$$

$$= \sum_{k=0}^{\infty} \text{Pr \{there are exactly k contacts in (0, t]\}} * \text{Pr \{the threshold is not crossed} < y\}$$

$$\therefore S(t) = \sum_{k=0}^{\infty} V_k(t)\, P\left[\sum_{i=1}^{k} x_i < y\right]$$

$$= \sum_{k=0}^{\infty} \frac{e^{-at}(at)^k}{k!}[[2g^*(\lambda)]^k - [g^*(2\lambda)]^k]$$

$$= 2\sum_{k=0}^{\infty} \frac{e^{-at}(at)^k}{k!}[g^*(\lambda)]^k - \sum_{k=0}^{\infty} \frac{e^{-at}(at)^k}{k!}[g^*(2\lambda)]^k$$

$$= 2\left[e^{-at} + e^{-at}\frac{atg^*(\lambda)}{1!} + e^{-at}\frac{(at)^2(g^*(\lambda))^2}{2!} + \ldots\ldots\right] - \left[e^{-at} + e^{-at}\frac{atg^*(2\lambda)}{1!} + e^{-at}\frac{(at)^2(g^*(2\lambda))^k}{k!} + \ldots\ldots\ldots\right]$$

$$= 2\,e^{-at}e^{at\,g^*(\lambda)} - e^{-at}e^{at\,g^*(2\lambda)}$$

$$\therefore S(t) = 2\,e^{-at}e^{at\,g^*(\lambda)} - e^{-at}e^{at\,g^*(2\lambda)}$$

$$L(t) = 1 - S(t)$$

$$= 1 - [2\,e^{-at}e^{at\,g^*(\lambda)} - e^{-at}e^{at\,g^*(2\lambda)}]$$

$$= 1 - [2\,e^{-at\,[1-g^*(\lambda)]} + e^{-at\,[1-g^*(2\lambda)]}]$$

Let g(.) ~ Exponential distribution with parameter μ

Let $g^*(\lambda) = \frac{\mu}{\mu+\lambda}$ and $g^*(2\lambda) = \frac{\mu}{\mu+2\lambda}$

$$L(t) = 1 - [\,2\,e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t} - e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}\,]$$

$$\Psi(t) = \frac{d}{dt}[L(t)]$$

$$= 2\left(\frac{a\lambda}{\mu+\lambda}\right)e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t} - \left(\frac{2a\lambda}{\mu+\lambda}\right)e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}$$

The expected time to seroconversion is given by

$$E(T) = \int_0^\infty t\Psi(t)\,dt$$

$$= \int_0^\infty t\left[\,2\left(\frac{a\lambda}{\mu+\lambda}\right)\mathrm{e}^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t} - \left(\frac{2a\lambda}{\mu+2\lambda}\right)\mathrm{e}^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}\,\right]\mathrm{dt}$$

$$= 2\left(\frac{a\lambda}{\mu+\lambda}\right)\int_0^\infty t\ \mathrm{d}\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right] - \left(\frac{2a\lambda}{\mu+2\lambda}\right)\int_0^\infty t\ \mathrm{d}\left[\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\right] \text{-------- (Eq.3)}$$

Consider

$$I_1 = \int_0^\infty t\ \mathrm{d}\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right]$$

$$I_2 = \int_0^\infty t\ \mathrm{d}\left[\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\right]$$

$$I_1 = \mathrm{t}\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right]_0^\infty - \int_0^\infty \frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\,\mathrm{dt}$$

$$= \frac{\mu+\lambda}{a\lambda}\int_0^\infty e^{-}\left(\frac{a\lambda}{\mu+\lambda}\right)\mathrm{dt}$$

on simplication, we get

$$= \frac{(\mu+\lambda)^2}{(a\lambda)^2} \text{ ------------ (Eq.4)}$$

$$I_2 = \mathrm{t}\left[\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\right]_0^\infty - \int_0^\infty \frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\,\mathrm{dt}$$

$$= \frac{\mu+2\lambda}{2a\lambda}\int_0^\infty e^{-}\left(\frac{2a\lambda}{\mu+2\lambda}\right)\mathrm{dt}$$

$$= \frac{(\mu+2\lambda)^2}{(2a\lambda)^2} \text{ ------------ (Eq.5)}$$

Substituting equation (4) and (5) in (3)

$$E(T) = 2\left(\frac{a\lambda}{\mu+\lambda}\right)\left(\frac{(\mu+\lambda)^2}{(a\lambda)^2}\right) - \left(\frac{2a\lambda}{\mu+2\lambda}\right)\left(\frac{(\mu+2\lambda)^2}{(2a\lambda)^2}\right)$$

$$= \frac{2(\mu+\lambda)}{a\lambda} - \frac{(\mu+2\lambda)}{2a\lambda}$$

$$= \frac{2[2(\mu+\lambda)]-(\mu+2\lambda)}{2a\lambda}$$

$$= \frac{4\mu+4\lambda-\mu-2\lambda}{2a\lambda}$$

$$E(T) = \frac{3\mu+2\lambda}{2a\lambda}$$

$$E(T^2) = \int_0^\infty t^2\Psi(t)\,dt$$

$$= \int_0^\infty t^2\left[2\left(\frac{a\lambda}{\mu+\lambda}\right)e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t} - \left(\frac{2a\lambda}{\mu+2\lambda}\right)e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}\right]dt$$

$$= 2\left(\frac{a\lambda}{\mu+\lambda}\right)\int_0^\infty t^2\, d\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right] - \left(\frac{2a\lambda}{\mu+2\lambda}\right)\int_0^\infty t^2\, d\left[\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\right] \text{-------- (Eq.6)}$$

Consider

$$I_1 = \int_0^\infty t^2\, d\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right]$$

$$I_2 = \int_0^\infty t^2\, d\left[\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\right]$$

$$I_1 = t^2\left[\frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\right]_0^\infty - \int_0^\infty \frac{e^{-\left(\frac{a\lambda}{\mu+\lambda}\right)t}}{-\left(\frac{a\lambda}{\mu+\lambda}\right)}\,dt$$

$$= \frac{2(\mu+\lambda)}{a\lambda}\int_0^\infty e^{-}\left(\frac{a\lambda}{\mu+\lambda}\right)\,dt$$

On simplication

$$= \frac{2(\mu+\lambda)^3}{(a\lambda)^3} \text{ ------------ (Eq.7)}$$

$$I_2 = \int_0^\infty t^2\, d\,\frac{e^{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)t}}{-\left(\frac{2a\lambda}{\mu+2\lambda}\right)}\Bigg]$$

On simplication

$$= \frac{2(\mu+2\lambda)^3}{(2a\lambda)^3} \text{ ------------ (Eq.8)}$$

Substituting equation (7) and (8) in (6)

$$E(T^2) = 2\left(\frac{a\lambda}{\mu+\lambda}\right)\frac{2(\mu+\lambda)^3}{(a\lambda)^3} - \left(\frac{2a\lambda}{\mu+2\lambda}\right)\frac{2(\mu+2\lambda)^3}{(2a\lambda)^3}$$

$$= \frac{4(\mu+\lambda)^2}{(a\lambda)^2} - \frac{2(\mu+2\lambda)^2}{(2a\lambda)^2}$$

$$= \frac{4\mu^2+4\lambda^2+8\mu\lambda}{a^2\lambda^2} - \frac{2\mu^2-8\lambda^2-8\mu\lambda}{4a^2\lambda^2}$$

$$= \frac{16\mu^2+16\lambda^2+32\mu\lambda-2\alpha^2-8\lambda^2-8\mu\lambda}{4a^2\lambda^2}$$

$$E(T^2) = \frac{14\mu^2+8\lambda^2+24\mu\lambda}{4a^2\lambda^2}$$

$$V(T) = E(T^2) - (E(T))^2$$

$$= \frac{14\mu^2+8\lambda^2+24\mu\lambda}{4a^2\lambda^2} - \left(\frac{3\mu+2\lambda}{2a\lambda}\right)^2$$

$$= \frac{14\mu^2+8\lambda^2+24\mu\lambda-9\mu^2-4\lambda^2-12\mu\lambda}{4a^2\lambda^2}$$

$$V(T) = \frac{5\mu^2+4\lambda^2+12\mu\lambda}{4a^2\lambda^2}$$

V.Numerical Illustration:

Table 1

	$\mu = 0.5$, λ= 1	
a	**Mean**	**Variance**
1	1.750	2.813
2	0.875	0.703
3	0.583	0.313
4	0.438	0.176
5	0.350	0.113
6	0.292	0.078
7	0.250	0.057
8	0.219	0.044
9	0.194	0.035
10	0.175	0.028

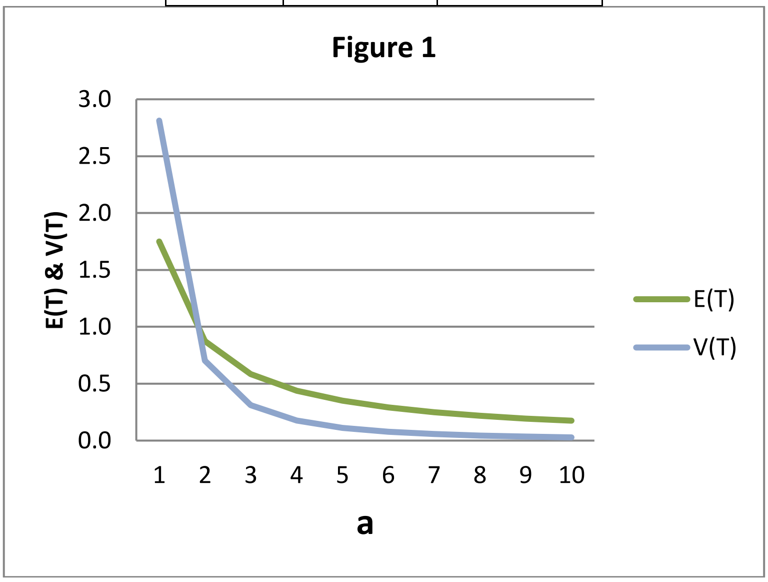


Table 2

	a= 2 , λ= 0.5	
μ	**Mean**	**Variance**
0.1	0.650	0.413
0.2	0.800	0.600
0.3	0.950	0.813
0.4	1.100	1.050
0.5	1.250	1.313
0.6	1.400	1.600
0.7	1.550	1.913
0.8	1.700	2.250
0.9	1.850	2.613
1	2.000	3.000

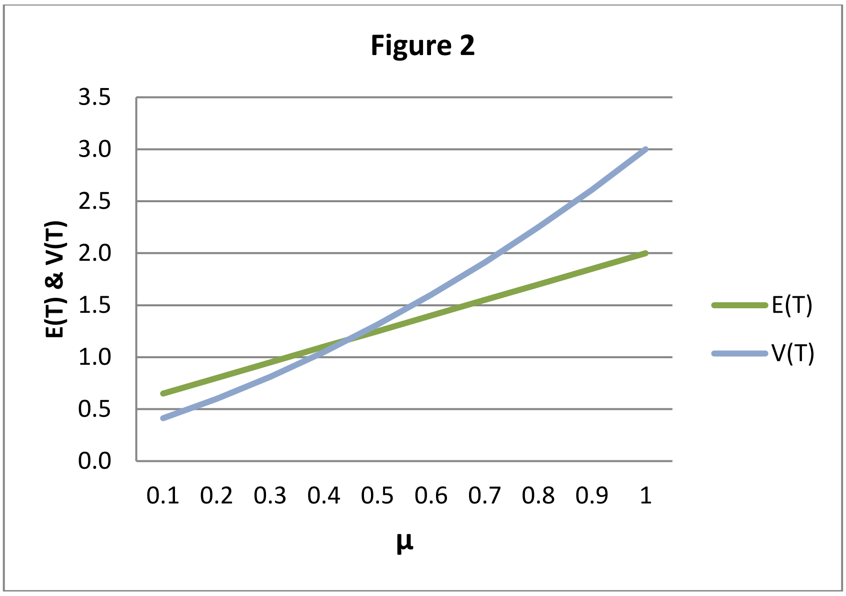
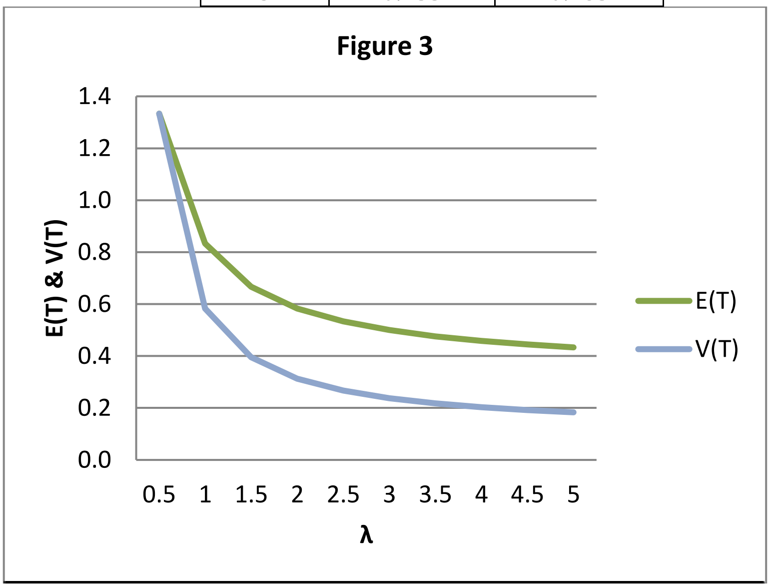


Table 3

	a= 3 , μ = 1= 0.5	
λ	**Mean**	**Variance**
0.5	1.333	1.333
1	0.833	0.583
1.5	0.667	0.395
2	0.583	0.313
2.5	0.533	0.267
3	0.500	0.238
3.5	0.476	0.218
4	0.458	0.203
4.5	0.444	0192
5	0.433	0.183



V. CONCLUSION

(i) If ‘a’ namely the parameter of the random variable denoting the contact rate increases then E(T) and V(T) decreases.

(ii) As the value of μ which is namely parameter of the random variable Xi denoting contribution to the antigenic diversity increases then it is seems that E(T) and V(T) are increases as indicated in table (2).

(iii) In table (3), the variation in E(T) and V(T) consequent to the change in the parameter λ is noted. As the parameter of the threshold distribution λ increases then E(T) as well as V(T) decreases.

REFERENCES

1. Anil, V.(2001). A Generalized Poisson distribution and its applications. Journal of the Kerala Statistical Association 12, 11-22.
2. Esary, J .D., Marshall, A.W. and Proschan, F. (1973). Shock Model and Wear Processes. Ann. Probability, 1(4), pp: 627-649.
3. Nowak, M.A., and R. M .May.(1991). Mathematical Biology of HIV Infections: Antigenic variation and Diversity Threshold. Mathematical Biosciences, Vol.106, pp: 1 – 21.
4. Stilianakis. N., Schenzle. D. and Dietz. K (1994). On the antigenic diversity threshold model for AIDS. Math. Bioscience, 121, pp: 235-247.
5. Kannan, R., Rabert, M., and Sathiyamoorthi, R. (2012). Estimation of expected time to cross the antigenic diversity threshold of HIV infected using mixed exponential distribution. Bio – Science Research Bulletin, Vol.28 (1), pp: 21- 32.
6. Kannan, R., and Iyappan, M. (2017). A Stochastic model approach determined the statistical measures for time to seroconversion of HIV infected using exponentiated modified Weibull distribution. International Journal of Applied Research, Vol. 3(6), pp: 1085- 1091.
7. Kannan, R., and Veena, A. (2022). A Shock Model approach for Seroconversion time under preventive strategy using Rayleigh distribution. High Technology Letters, Vol.28 (10), pp: 283-294