Modified Equation for Speed of Efflux

Kattubadi Drutesh

Student
Electronics and Communication Department,
GITAM School of Technology, Bangalore.

Abstract: Random errors can occur due miscalculations, which may lead to fraction of changes in the end results. Through these changes, the accurate value can be changes resulting in false data most of the time. To get an accurate value, the calculation must be cross checked precisely. The speed of any liquid which escapes through a hole from a container is measured using the Speed of Efflux equation which is implemented using Torricelli's law in addition to Bernoulli's Principle. But there exists a miscalculation in the derivation of the equation for speed of efflux, regarding the draining velocity of the liquid in the container. This absence of a term in the final equation may lead to many random errors while measuring the speed of efflux. This paper deals with those errors in the derivation of the speed of efflux and modifies the final equation which is similar to the equation of the Torricelli's theorem. This modified equation for speed of efflux which is similar to that of Torricelli's equation results in accurate values for the velocity of the liquid escaping out of the container through a hole. Through these many random errors found in real time applications which use fluid mechanics, can be over taken and desired accurate values can be gained.

Keywords: Torricelli's law, Bernoulli's Principle, Speed of Efflux, discharge velocity.

INTRODUCTION:

In relation to the speed of a fluid emerging through an aperture, Torricelli established a law, which had been later demonstrated as a specific application of the Bernoulli's principle. He discovered, the rate at which water escapes from a small hole in a container's bottom is related to square root of water's depth from the top of the surface to the hole, opening, or reference point. The velocity of efflux is the rate at which a liquid will emerge from a hole drilled into the bottom of a container filled with liquid.

According to Torricelli's law, a liquid's discharge velocity is equal to its free fall from its surface to the tank's opening. Under the assumption that there doesn't exists any air resistance or any viscosity, or any other obstruction to the flow of the fluid, this law focuses on the departing velocity of the jet of the liquid dependent on the depth below the surface of the liquid at where the outlet starts.

This law is best demonstrated by the "spouting can", a liquid contained in a container with orifices drilled at various heights. Liquid level above each orifice controls the discharging velocity across each orifice as shown in fig.1.

But there exists a miscalculation in the derivation of the equation for speed of efflux, regarding the draining velocity of the liquid in the container. This paper deals with the correction in the derivation of the speed of efflux in reference with the Torricelli's formula. Through this equation much more accurate values of velocity of the liquid escaping out of the container can be obtained. These equations can further be used to implement ordinary differential equations, speed of efflux for various viscous liquids, and many more.

In this paper, we derive an equation for speed of efflux considering the drain velocity of the fluid in the container, in reference with the Torricelli's theorem.

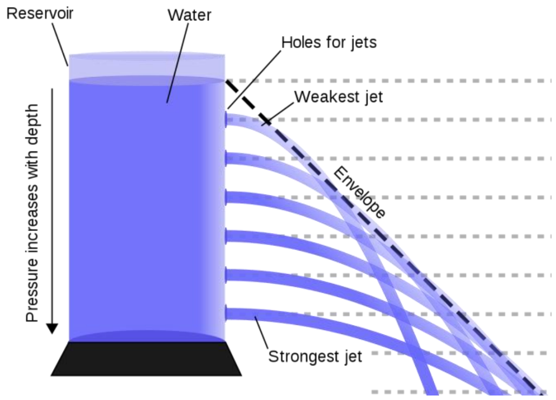

Fig.1: Spouting can

EXISTING METHODS:

Torricelli's law is a concept of fluid dynamics that links the rate of fluid flow from an orifice to the height of fluid above the opening. Though not in this form, the law was discovered in 1643 by Torricelli. In the end, it was determined that it was an application of Bernoulli's principle [1].According to the law, a body/drop of water will fall from a height of h with a speed of v through an orifice at the bottom of the tank filled to that depth, or $v^2 = 2gh$.

Torricelli Theorem:

First of all let us consider the Torricelli's theorem for reference. Evangelista Torricelli developed the Torricelli's equation, sometimes known as the Torricelli formula, to determine the end velocity of an object travelling down an axis at a constant acceleration without a specified time interval.

Let

- Δx - change in position of object in m (meters)
- v_f - final velocity of the object in m/s
- v_i - initial velocity of object in m/s
- a - acceleration of object in m/s^2 and it is constant

The equation is:

$$(v_f)^2 = (v_i)^2 + 2a\Delta x \text{ [2] [3]}$$

The derivation for Torricelli's Theorem is:

the definition of acceleration;

$$a = (v_f - v_i)/\Delta t$$

where Δt is the time interval.

For the final velocity :

$$V_f = v_i + a\,\Delta t$$

Square both sides to get:

$$(v_f)^2 = (v_i + a\,\Delta t)^2 = (v_i)^2 + 2a\,v_i\,\Delta t + a^2(\Delta t)^2 \qquad (A)$$

The term $(\Delta t)^2$ in this equation at end of the water drop/object travelling with an acceleration which is constant, to be isolated, also exists in other equation that is appropriate for the motion:

$$x_2 = x_1 + v_1\,\Delta t + a(\Delta t)^2/2$$

$$x_2 - x_1 - v_1\,\Delta t = a(\Delta t)^2/2$$

$$(\Delta t)^2 = 2[x_2 - x_1 - v_i\Delta t]/a = 2[\Delta x - v_i\Delta t]/a \qquad (B)$$

Substitute Equation B into the Equation A:

$$(v_f)^2 = (v_i)^2 + 2a\,v_i\,\Delta t + a^2(2[\Delta x - v_i\Delta t]/a)$$

$$(v_f)^2 = (v_i)^2 + 2a\,v_i\,\Delta t + 2a(\Delta x - v_i\Delta t)$$

$$(v_f)^2 = (v_i)^2 + 2a\,v_i\Delta t + 2a\,\Delta x - 2a\,v_i\Delta t$$

Therefore,

$$(v_f)^2 = (v_i)^2 + 2a\Delta x \text{ [2][3]} \qquad (C)$$

Equation C is a standard version of Torricelli's Equation. A constant acceleration on any axis is valid for this equation to be applied.

Thus, the Torricelli theorem is derived and proved. This equation is used as the reference to check whether the equation for speed of efflux is similar the equation of Torricelli theorem.

Speed of efflux using Torricelli's Theorem:

According to Torricelli's law for Newtonian fluids, the speed that a body would reach in a freely falling situation when falling from a height h is the same as the amount density of efflux of the fluid passing through an orifice at the walls of the container filled with the liquid at a depth of h[4].

In relation to the speed of a fluid emerging through an aperture, Torricelli also found Torricelli's law, which was later revealed to be a specific application of Bernoulli's principle. Torricelli discovered that the pace at which water escapes from a small hole in a container's bottom is directly proportional to square root of the water's depth [5]. Since y represents the water's depth at time t and the cylindrical container is with an orifice at the bottom, then

$$dy/dt = -k\sqrt{(u(y).y)}$$

this equation is further used in implementation of Elementary ordinary differential equations [6].

The law illustrates the relation between the height of the liquid inside the container and the amount of fluid that leaks out. The relationship can be summed up in the way that follows. If the open container is filled with fluid has a tiny hole at the bottom, through which the fluid passes at the same rate as if it is poured from the same height as the orifice level. The fluid would have a velocity v if it were dropped from that height, and v will be of equal velocity at which the fluid would exit the hole if the fluid's height h were the same as the liquid dropped into the container [7].Then further we suppose that this fluid is to be perfect, which implies the flow is luminary, compressible, & without viscosity [8]. These factors cannot be disregarded because the viscosity and flow of non-fluids may differ from those of liquids, making it impossible to apply the same principles to them.

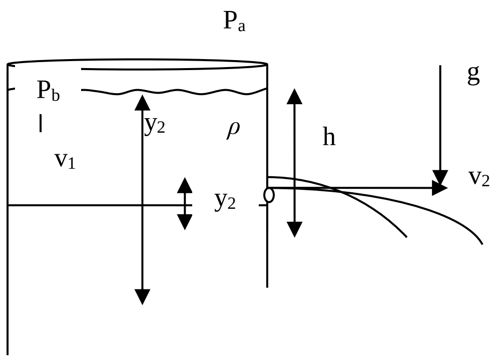

Fig.2: Speed of Efflux

Derivation of speed of efflux in reference with the Torricelli's Theorem:
Under the required conditions, the Bernoulli's principle states that;

$V^2/2 + gh\ P/\rho$ = constant

Where,

- v - velocity of the liquid in m/s
- g - acceleration due to gravity in m/s^2
- h - height of the liquid above the outlet point
- ρ - density in kg/m^3
- P - atmospheric pressure on top of the tank in N/m^2

Think of a container as shown in fig.2, holding a liquid with density and a small hole in its side at a height y_1 from the bottom. The liquid's surface is at height y_2, and the air above it has a pressure of P.
We know that from the equation of continuity,

$$A_1v_1 = A_2v_2 \quad \text{or} \quad v_2 = A_1v_1/A_2 \quad (1)$$

If the tank's cross-sectional area is significantly greater than the hole's (A_2>>A_1), we can assume that the fluid is roughly at rest at the top, that is $v_1 = 0$ and consider $P_b = P_a$ (the atmospheric pressure). Now applying the Bernoulli equation at the hole, we have

$$P_a + (½)\rho v_1^2 + \rho g y_1 = P_b + (½)\rho v_2^2 + \rho g y_2 \quad (2)$$

When the aperture is incredibly small compared to the cylindrical cross section of the container, it is assumed that the surface's drain velocity is minimal[9].

$$v_1=0 \quad (3)$$

Let us consider the height = h = y_1-y_2 as shown in Fig.2
From above mentioned Equations 1 , 2 and 3, we have

$(½)\rho v_2^2 = P_a - P_b + \rho g y_1 - \rho g y_2$

$v_2^2 = (2/\rho)[(P_a - P_b) + \rho g(y_1 - y_2)$

$v_2 = \sqrt{}\ [(P_a - P_b) + 2gh]$

If tank is opened at the top $P_b = P_a$
therefore speed of efflux is ,

$$v_2 = \sqrt{2gh} \text{ or } (v_2)^2 = 2gh \quad (4)$$

This Equation 4 is the generic formula used for calculating the speed of efflux.

MODIFIED EQUATION:

As the liquid is draining out, the level of the liquid gradually drops which leads to the establishment of not only the speed of efflux but also the initial speed of the surface of the liquid.

Considering Equation 3, when the surface of the liquid layer lowers to a certain depth after some time, $v_1 = 0$ doesn't apply because, irrespective of the size of the hole compared to the container. From the fig.2 we can observe that the liquid has covered a certain distance or depth from its initial surface position towards the hole while leaking, consuming some amount of time. This relates the depth covered and time consumed from the initial surface of the liquid to the rise of its velocity called as drain velocity (v_1) [10]. Let us consider v_1 not to be equal to 0, we get

$$v_1 \neq 0 \quad (5)$$

$P_a + (½)\rho v_1^2 + \rho g y_1 = P_b + (½)\rho v_2^2 + \rho g y_2$

From above mentioned equations 1 , 2 and 4, we have

$(½)\rho v_2^2 = P_a - P_b + (½)\rho v_1^2 + \rho g y_1 - \rho g y_2$

$v_2^2 = (2/\rho)[(P_a - P_b) + \rho g(y_1 - y_2) + (½)\rho v_1^2$

$v_2 = \sqrt{}\ [(P_a - P_b) + 2gh + (v_1)^2]$

If tank is opened at the top $P_b = P_a$

$$v_2 = \sqrt{\{(v_1)^2+2gh\}}$$

Therefore,

$$(v_2)^2 = (v_1)^2+2gh \quad (6)$$

Equation (5) is the modified equation for speed of efflux, which now considers the factors of draining velocity in the container and the height at which the orifice is placed below the liquid surface in the container.

RESULTS:

Equation 4 & Equation 6 are cross checked with the Equation C and following results are tabulated below in Table 1.

Table 1: Equations for Speed of efflux in reference with Torricelli's equation

Torricelli's equation	***Speed of efflux equation***	***Modified equation for speed of efflux***
$(v_f)^2 = (v_i)^2 + 2a\Delta x$	$(v_2)^2 = 2gh$	$(v_2)^2 = (v_1)^2 + 2gh$

If we clearly observe the Table 1, the modified equation for speed of efflux possessing the draining velocity of the liquid in the container is similar in terms to that of the Torricelli's equation. Through this a modified equation for speed of efflux is proved. The speed of efflux (v_2) in the modified equation represents the final velocity (v_f), the draining velocity of the liquid from the surface (v_2) is similar to that of initial velocity (v_i) and the term 2gh is the term which is same as that of $2a\Delta x$ in Torricelli's equation. Through these similarities we can conclude that the modified equation for speed of efflux is similar to that of Torricelli's equation.

CONCLUSION:

The law of Torricelli has real-world applications. This physical law outlines a significant link between the liquid's height inside the container, its drain velocity from the surface and its departure velocity from an orifice. By using this Torricelli's theory, we understood this relationship and how to determine exit velocity.

This paper observes the miscalculations in generic equation for speed of efflux regarding the draining velocity of the liquid from the surface and modifies it as per the Torricelli's theorem, by adding the required draining velocity of the liquid from the surface to the generic equation, so that it resembles close similarity with the Torricelli's equation. Even though the only difference in the generic equation and the modified equation for the speed of efflux is the term with draining velocity component, this modified equation stays similar to that of Torricelli's equation. Through this modified equation for speed of efflux in reference with Torricelli's equation, accurate values for velocity of the liquid at the departure can be calculated with lesser error rates in real time applications of fluid mechanics.

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