An Analysis of Properties in \( L(\text{Open} F_{\sigma}, \text{Open}) \) Functions in Topological Spaces

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Abstract: \( D \)-Continuous mappings prompted us to study a new class of functions namely \( L(\text{Open} F_{\sigma}, \text{Open}) \) functions which contains properly the class of totally continuous mappings and is contained in the class of continuous mappings. A few properties of these functions are discussed in this paper.

Keywords: Continuous mappings, Open \( F_{\sigma} \), Inverse image, Domain, Hausdorff space, \( D \)-regular space.

1. Introduction:
According to Noiri and Yuksel [1], [2] define various functions from a topological space \( X \) to another topological space \( Y \) have been introduced so far. These functions are continuous, non-continuous, weak continuous, strong continuous by various authors and researchers from time to time under different designations. A mapping \( f : X \rightarrow Y \) is said to be \( D \)-continuous if inverse image of every open \( F_{\sigma} \) set is open. Hamlett and Jankovic [5] showed that the collection of all open \( F_{\sigma} \) sets in a space constitutes a base for a weaker topology and both the topologies coincide if the space is \( D \)-regular and hence continuous mappings with range space \( D \)-regular constitute a class which is the same as that of \( D \)-continuous mappings defined by Kohli [4] in the same reference, we study here, \( L(\text{Open} F_{\sigma}, \text{Open}) \) functions using the nomenclature for this we have to study \( F_{\sigma} \) sets which share the countable union of closed sets in a topological space. G. Aslim, A. Caksu Guler, and T. Noiri [6] study new functions are obviously a stronger form of continuity by means of open \( F_{\sigma} \) sets which coincides with continuity if the domain space is a \( D \)-regular countable space. Actually in a \( D \)-regular space every open set is union of open \( F_{\sigma} \) sets and countability of the space gives it is countable union of open \( F_{\sigma} \) sets and hence open \( F_{\sigma} \) set it. Thus every open set is open \( F_{\sigma} \) set in a \( D \)-regular countable space.

2. Definitions and Characterizations:

Definition 2.1:
A function \( f : X \rightarrow Y \) is said to be \( L(\text{Open} F_{\sigma}, \text{Open}) \) at \( x \in X \) if for each open set \( V \) containing \( f(x) \) there exists an open \( F_{\sigma} \) set \( U \) containing \( x \), such that \( F(U) \subseteq V \) and \( f \) is called \( L(\text{Open} F_{\sigma}, \text{Open}) \) if it is \( L(\text{Open} F_{\sigma}, \text{Open}) \) at each \( x \) in \( X \).

Theorem 2.1: For a mapping \( f : X \rightarrow Y \), the following are equivalent conditions provided \( X \) is a countable space.

a) \( f \) is \( L(\text{Open} F_{\sigma}, \text{Open}) \)
b) Inverse image of every member of a base is an open \( F_{\sigma} \) set.

Proof: Let \( V \) be a member of a base for \( Y \), and \( x \in f^{-1}(V) \) or. \( x \in V \) so there exists an open \( F_{\sigma} \) set \( U \) containing \( x \) such that \( f(U) \subseteq V \), \( x \in U \subseteq f^{-1}(V) \) thus \( f^{-1}(V) \) is an open \( F_{\sigma} \) set.

Hence \( a \Rightarrow b \)

There is the following porism according to above theorem:

a) Inverse image of every open set is an open \( F_{\sigma} \) set.
b) Inverse image of every closed set is closed \( G_{\delta} \) set.
c) For each \( x \in X \) and each net \( (x_{\alpha}), \alpha \in D \) which is eventually, in each open \( F_{\sigma} \) set containing \( x \), the net \( f(x_{\alpha}) \),
\[ \alpha \in D \] converges to \( f(x) \)

d) For each \( x \in X \) and each filter base \( \beta \in B_\alpha \), for which each open \( F_\sigma \) set \( V \) there exists \( B_\beta \in \beta \) such that \( B_\beta \subset V \), \( f(\beta) \) converges to \( f(x) \)

3. Comparison and analysis of the Properties:

According to Yüksel, Açıkgoz and Noiri [13] define that every cl-open set is \( F_\sigma \), so every totally continuous [5] function is \( L(\text{Open}F_\sigma, \text{Open}) \) but the converse is not true.

1. The identity map on \((R, U)\), the real line is \( L(\text{Open}F_\sigma, \text{Open}) \) but not totally continuous defined by Hitir, and T. Noiri [10], [11] every \( L(\text{Open}F_\sigma, \text{Open}) \) mapping is continuous but the converse is not true as.

2. The identity map on \( X = \{a, b\} \) with \( T = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \) is super continuous [9] and hence continuous but not \( L(\text{Open}F_\sigma, \text{Open}) \).

E. Hitir, A. Keskin, and T. Noiri [12] show that \( L(\text{Open}F_\sigma, \text{Open}) \) a map is independent of completely continuous mappings and \( \beta \) - continuous mappings.

3. Let \( R \) be the usual space of real’s and \( Y = \{a, b, c, d\} \) with
\[ T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\} \]

Define a map \( g : R \rightarrow Y \) by
\[ g(x) = \begin{cases} 
  a; x < p \\
  b; p < x < q \\
  c; q < x < r \\
  d; x > r 
\end{cases} \]

Where \( p, q \) & \( r \) are distinct and real’s, then \( g \) is \( L(\text{Open}F_\sigma, \text{Open}) \) but not \( \beta \) -continuous and hence not completely continuous as shown in [13] nor homeomorphism.

Let \( X = \{a, b, c, d\} \) with \( T = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c, d\}\} \) and \( Y = \{p, q, r\} \) with \( U = \{\emptyset, \{p\}, Y\} \)

Define a mapping \( f : X \rightarrow Y \) by
\[ f(x) = \begin{cases} 
  f(a) = f(b) = p \\
  f(c) = q \\
  f(d) = r 
\end{cases} \]

Then \( f \) is completely continuous, \( \beta \) -continuous but not \( L(\text{Open}F_\sigma, \text{Open}) \) its shows that even a homeomorphism may fail to be \( L(\text{Open}F_\sigma, \text{Open}) \).

4. Basic Properties:

For \( L(\text{Open}F_\sigma, \text{Open}) \) mappings the following theorem has some trivial results.

If \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) is

1. Continuous whenever \( f \) is \( D \) – continuous and \( g \) is \( L(\text{Open}F_\sigma, \text{Open}) \) mapping.
2. \( L(\text{Open}F_\sigma, \text{Open}) \) Mapping whenever \( f \) is \( L(\text{Open}F_\sigma, \text{Open}) \) and \( g \) is continuous.
3. Composition of two \( L(\text{Open}F_\sigma, \text{Open}) \) mappings is \( L(\text{Open}F_\sigma, \text{Open}) \) mapping.
4. \( P_a \) \( f : X \rightarrow X_a \) is \( L \) (Open \( F_\sigma \), Open) iff \( f \) is \( L(\text{Open}F_\sigma, \text{Open}) \) where \( P_a \) is \( \alpha \)th projection of the product space onto \( X_a \).
Theorem 4.1 If \( f : X \rightarrow Y \) is a surjection carrying open \( F_\sigma \) (or closed \( G_\delta \)) sets onto open (closed) sets and, \( g : Y \rightarrow Z \) is any mapping such that \( gof \) is \( L(OpenF_\sigma, Open) \) then \( g \) is continuous.

**Proof:** If \( U \) is an open (or Closed) set in \( Z \), then \( (gof)^{-1}U = f^{-1}\{g^{-1}(U)\} \) is open \( F_\sigma \) (or closed \( G_\delta \)) set in \( X \) and, hence \( f^{-1}\{f^{-1}\{g^{-1}(U)\}\} = g^{-1}(U) \) is open (closed) in \( Y \).

**Corollary 4.1(1):** Let \( f : X \rightarrow Y \) be \( L(OpenF_\sigma, Open) \) and \( A \subseteq X \) If \( G \) is open in \( Y \), then \( f^{-1}(G) \) is open - \( F_\sigma \) in \( X \).

**Explanation:** Since \( A \) is \( F_\sigma \) in \( B \) so \( A=UB_i \) where each \( B_i \) is closed in \( B \).

Also, \( B_i = X_i \) intersection \( B \) where each \( X_i \) is closed in \( X \).

Thus, \( A \) = union of \( X_i \) intersection, \( B = \bigcup X_i \) intersection \( B \) is a \( F_\sigma \) set in \( X \) being intersection of two \( F_\sigma \) sets.Openness of \( A \) is obvious in \( X \).

Noiri and Umehara [9] notified that if \( A \) is closed \( G_\delta \) set in \( B \) and \( B \) is closed \( G_\delta \) set in \( X \) then \( A \) is closed \( G_\delta \) set in \( X \).

**Theorem 4.2** Let \( X = A \) union \( B \), where \( A \) and \( B \) are open \( F_\sigma \) sets in \( X \), and \( f : A \rightarrow Y \), \( g : B \rightarrow Y \) be \( L(OpenF_\sigma, Open) \) functions.

If \( f(x) = g(x) \) for every \( x \) in \( A \cap B \), then \( h : X \rightarrow Y \), defined by

\[
h(x) = \begin{cases} 
  f(x) & \text{if } x \in A \\
  g(x) & \text{if } x \in B 
\end{cases}
\]

is \( L(OpenF_\sigma, Open) \).

**Proof:** Since \( h^{-1}(U) = f^{-1}(U) \) union \( g^{-1}(U) \) for every open set \( U \) in \( Y \), therefore \( h^{-1}(U) \) is open is open \( F_\sigma \) as \( f^{-1}(U) \) and \( g^{-1}(U) \) are both open \( F_\sigma \) in open \( F_\sigma \) sets \( A \) and \( B \) respectively and hence the same in \( X \).

**Corollary 4.2(1)** If \( A \) and \( B \) are closed \( G_\delta \) sets in \( X \), in instead of open \( F_\sigma \) sets, the theorem remains unaltered.

**Corollary 4.2(2)** If \( X = U \) \( (U_\alpha : \alpha \in A) \) where \( U_\alpha \) are open \( F_\sigma \) and pairwise disjoint sets in \( X \), such that \( f_\alpha : U_\alpha \rightarrow Y \) is \( L(OpenF_\sigma, Open) \) for each \( \alpha \).

Then \( h : X \rightarrow Y \) defined by \( h(x) = f_\alpha(x) \) if \( x \in U_\alpha \) is a \( L(OpenF_\sigma, Open) \) Function.

**Theorem 4.3** Let \( f \) and \( g \) be \( L(OpenF_\sigma, Open) \) function from a space \( X \) into a \( T_2 \)-space \( Y \) then, \( A = \{ x : f(x) = g(x) \} \) is closed \( G_\delta \) in \( X \), provided \( A \) is Co-countable.

**Proof:** For each \( x \in X - A \), we can show that the existence of open \( F_\sigma \) set \( G : x \in G \subset X - A \).

Thus, \( X - A \) is countable union of \( F_\sigma \) sets and hence, \( A \) is closed \( G_\delta \).

Condition: If \( f \) and \( g \) agree on a co-countable set \( B \) such that smallest closed \( G_\delta \) set containing \( B \) is \( X \). Then \( f = g \).

A space \( X \) is called \( D \)-regular [14] if for each \( x \in X \) and each open set \( U \) containing \( x \), there exists an open \( F_\sigma \) set \( V \) such that
\[ x \in V \subseteq U . \] Obviously, every open set in a \( D \)-regular space is union of open \( F_\sigma \) sets and hence, will be open \( F_\sigma \) itself provided it is countable.

**Corollary 4.3(1)** Let \((X, T)\) be a topological space, the following statements are equivalent

a. \((X, T)\) is a \( D \)-regular and countable space (i.e., set \( X \) is countable).

b. Every continuous functions \( f \) from \( X \) into a topological space \( Y \) in \( L(\text{Open} F_\sigma, \text{Open}) \) function.

**Corollary 4.3(2)** The set of fixed points of an \( L(\text{Open} F_\sigma, \text{Open}) \) function on a Hausdorff defined by Ajmal and Kohli \[8,7\] and \( D \)-regular and countable space \( X \) is a closed \( G_\delta \) set, where \( A = \{ x : x \in f(x) \} \) Identity map on \( X \) is obviously \( L(\text{Open} F_\sigma, \text{Open}) \) in view of Continuous functions. Thus \( f \) and identity map are fulfilling so \( A \) is closed \( G_\delta \) set.

**5 Conclusion and Future Work:**

Every open set is \( F_\sigma \) in a perfectly normal space and a space is perfect if every open (or closed) set is \( F_\sigma \) (or \( G_\delta \) set), so every continuous function from a perfect space is \( L(\text{Open} F_\sigma, \text{Open}) \) introduced a function \( f : X \to Y \) to be \( z \)-continuous if for each \( x \in X \) and each co-zero set \( V \) containing \( f(x) \), there is an open set \( U \) containing \( x \) such that \( f(U) \subseteq V \) for every \( D \)-Continuous function is \( z \)-continuous but the converse fails. We have that every \( z \)-continuous function is a \( D \)-Continuous provider where the range space is normal. Since in a normal space every closed \( G_\delta \) set is a zero set or every open \( F_\sigma \) set is a co-zero set. The following conclusion is trivial:

i. If \( X \) is a countable set and zero-dimensional space then every continuous mapping from \( X \) \( L(\text{Open} F_\sigma, \text{Open}) \)

ii. For each \( \alpha \in I \), let \( f_\alpha : X_\alpha \to Y_\alpha \) be a mapping and let \( f_\alpha : \pi X_\alpha \to \pi Y_\alpha \) be defined as \( f(\alpha \alpha) = f_\alpha (x_\alpha) \) for each \((\alpha \alpha)\) in \( \pi X_\alpha \).

**Conflict of Interest:**

On behalf of all authors, the corresponding author states that there is no conflict of interest.

**References:**


