Modelling Effect Of The Depleting Dissolved Oxygen On The Existence Of Intracting Planktonic Population With Nutrient Cycling

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Abstract : The effect of dissolved oxygen depletion on the existence of interacting planktonic populations with nutrient cycling is investigated in this work using a mathematical model. A system of non-linear ordinary differential equations is used to formulate the mathematical model. Nutrient content, algal density, Zooplankton population density and dissolved oxygen concentration are among the four state variables included in the model. The requirements for the presence of the internal equilibrium are specified after obtaining all of the system's possible equilibria. All possible equilibrium points are subjected to a local stability study. The non linear stability analysis of the non trivial equilibrium point was performed, and numerical simulation was used to determine the criterion for the species survival or extinction.

Keywords : Nutrient, Dissolved oxygen, Algal density, stability, Zooplankton, Liapunov Function.

(1) INTRODUCTION

Phosphorus and nitrogen are all around us, and more importantly around lake. Since most plant life, aquatic, rely heavily on both, they all contain large amounts of nutrients. These nutrients consist of lawn fertilizers, garden / flower bed fertilizers, farm fields and pastures, and wild life. When golf fairways and greens, lawns, gardens, flower beds, and farm fields are fertilized, there is an often large amounts of excessive nutrients that are not used quick enough by the terrestrial plants or that the soil cannot hold. When the first rain or irrigation comes along in these areas, those extra amounts of nitrogen and phosphorous "run off" into the low areas. Most lakes are the lowest spots in a given area because they need to be able to hold water. Therefore, the nutrients that runoff are flowing directly into the lake.

Due to excessive growth of macrophytes in water and algae floating on the water surface, the photosynthesis process of aquatic flora decrease leading to decreased production of oxygen in the water body. Due to the oxygen deficit, growth rate of many aquatic populations decrease [13] and the habitat also deteriorates due to the decrease in the level of transparency of the aquatic population has been investigated by Chaturvedi and Misra [1].

Several investigators have studied the depletion of dissolved oxygen and survival of aquatic population in a lake due to presence of algae and zooplankton [4, 5, 6, 7, 9]. Voinov and Tonkikh [3] have presented an eutrophication model in an unpolluted lake assuming that the nutrient is supplied only by detritus of algae and macrophytes and have not considered the discharge of nutrients by water runoff from agricultural fields. Jayaweera et.al. [8] studied bio-manipulation in shallow eutrophic lakes by using a mathematical model involving phytoplankton, zooplankton, detritus, bacteria and fish population but they did not consider the supply of nutrients from outside. Shukla et. al. [7] presented a non-linear mathematical model for the depletion of dissolved oxygen in a lake caused by algal bloom, but in this model, we have not considered, macrophytes role on the depletion of dissolved oxygen. Misra [2, 4, 5] studied the depletion of dissolved oxygen in a lake due to submerged macrophytes. But in this model, they have not considered the effect of oxygen deficit on the plankontic population.

Many researchers [9, 10, 11, 12] have studied the nutrient, phytoplankton, zooplankton system with nutrient cycling. Khare et. al. [10] have studied the role of toxin producing phytoplankton on a plankton ecosystem. In this chapter, a mathematical model has been proposed to study the depletion of dissolved oxygen in plankton ecosystem with nutrient cycling.

Keeping in view of the above, in this chapter, we have study the effect of the depleting dissolved oxygen on the existence of intracting planktonic population with nutrient cycling.

(2) MATHEMATICAL MODEL

In this chapter, we consider a waterbody, where the eutrophication process is governed by nutrients, algae, plankontic and concentration of dissolved oxygen. Let n be the cumulative concentration of various nutrients, a be the cumulative density of algae, P be the density of plankontic and C be the concentration of dissolved oxygen. We assume that the cumulative rate of discharge of nutrients into the waterbody is q, which is depleted with rate $\alpha_o n$. It is further assumed that the growth rate of nutrients by algae is $(\pi_1\alpha_1 a)$. The depletion of cumulative concentration of nutrients by algae is proportional to the monod interaction of the density of algae and to the concentration of nutrient (i.e. $\beta_1 na/(\beta_{12}+\beta_{11}n)$). Thus, the growth rate of algae, which is assumed to be wholly dependent on the nutrients, is proportional to this fraction. The depletion rate of algae and zooplankton are α_1, r_1 respectively. α_3 is rate of predation of algae by zooplankton. It is consider that the rate of growth of dissolved oxygen through air-water interaction is q_c assumed to be a constant and α_2 is natural depletion rate of concentration C. It is further assumed that, the rate of depletion of dissolved oxygen by algae is proportional to a (i.e. $\beta\alpha_1 a$). The growth rate of plankontic by algae is proportional to the terms (i.e. $\pi_2\alpha_3 aP/(\beta_2 + C_0 - C)$). β_{12}, β_2 are half saturation constants, C_o is DO saturation value and C_o- C is oxygen deficit. π_1 is the fraction of dead algae population that is being recycled back to the nutrient pool. In view of the above considerations, the system is governed by the following differential equations:-

$$\frac{dn}{dt} = q - \alpha_o n - \frac{\beta_1 na}{(\beta_{12} + \beta_{11} n)} + \pi_1 \alpha_1 a \quad (1)$$

$$\frac{da}{dt} = \frac{\theta_1 \beta_1 na}{(\beta_{12} + \beta_{11} n)} - \alpha_1 a - \alpha_3 aP \quad (2)$$

$$\frac{dC}{dt} = q_c - \alpha_2 C - \beta \alpha_1 a \quad (3)$$

$$\frac{dP}{dt} = \frac{\pi_2 \alpha_3 aP}{\beta_2 + C_0 - C} - r_1 P \quad (4)$$

With initial conditions $n(0) = n_{10} > 0$, $a(0) = a_{10} > 0$, $C(0) = C_{10} > 0$, $P(0) = P_{10} > 0$. Where q, α_0, β_1, β_{12}, β_{11}, θ_1, α_3, q_c, α_2, β, α_1, π_2, α_3, β_2, C_0, r_1 and $0 < \pi_1 < 1$ are positive constants.

(3) BOUNDEDNESS AND EQUILIBRIA OF THE SYSTEM

In this section, we analyze the system of equations (1) – (4) under the initial conditions $n(0) = n_{10} > 0$, $a(0) = a_{10} > 0$, $C(0) = C_{10} > 0$ and $P(0) = P_{10} > 0$. In the following lemma we have shown that all the solutions are bounded in the region Ω_1.

Lemma (1): The set

$\Omega_1 = \{(n, a, C, P) \in R_4^+ : 0 \le n + a + P \le \frac{q}{\delta_m}, 0 \le C \le R_c\}$ is a region of attraction for all solutions initiating in the interior of positive octant, where

$\delta_m = \text{Min }\{\alpha_0, (1-\pi_1)\alpha_1, r_1\}$ and $R_c = \frac{q_c}{\alpha_2}$

Proof:- Let us consider the following function:

$w(t) = n(t) + a(t) + P(t)$

$$\frac{dw}{dt} = \frac{dn}{dt} + \frac{da}{dt} + \frac{dP}{dt}$$

from model (1) – (4) and if $\delta_m = \min(\alpha_0, (1-\pi_1)\alpha_1, r_1)$, then we obtain the following expression:-

$$\frac{dw(t)}{dt} \le q - \delta_m w(t),$$

$$\frac{dw(t)}{dt} + \delta_m w(t) \le q,$$

Now, applying the theorem of differential inequalities we obtain

$$w(t) \le w(0)\ e^{-\delta_m t} + \frac{q}{\delta_m},$$

As $t \to \infty$, we have

$$0 \le w(t) \le \frac{q}{\delta_m},$$

$$0 \le n + a + P \le \frac{q}{\delta_m},$$

from equation (3.2.3), we have

$$\frac{dC}{dt} = q_c - \alpha_2 C - \beta \alpha_1 a,$$

$$\frac{dC}{dt} + \alpha_2 C = q_c - \beta \alpha_1 a,$$

$$\frac{dC}{dt} + \alpha_2 C \le q_c,$$

This is liner equation of first order, we get,

$$c(t) \le \frac{q_c}{\alpha_2},$$

Hence, the solution of the system (1) – (4) is bounded in Ω_1.

The model (1) – (4) has three non-negative equilibria, namely

(i) $E_1\left(\dfrac{q}{\alpha_0}, 0, \dfrac{q_c}{\alpha_2}, 0\right)$

(ii) $E_2(\bar{n}, \bar{a}, \bar{C}, o)$, Where

$$\bar{n} = \frac{\alpha_1\beta_{12}}{(\theta_1\beta_1 - \alpha_1\beta_{11})},$$

$$\bar{a} = \frac{\theta_1}{\alpha_1(1-\theta_1\pi_1)}\left[\frac{q(\theta_1\beta_1 - \alpha_1\beta_{11}) - \alpha_0\alpha_1\beta_{12}}{(\theta_1\beta_1 - \alpha_1\beta_{11})}\right],$$

$$\bar{C} = \frac{q_c - \beta\alpha_1\bar{a}}{\alpha_2},$$

Thus, E_2 exist if

$\theta_1\beta_1 - \alpha_1\beta_{11} > 0$, $(\theta_1\beta_1 - \alpha_1\beta_{11})q - \alpha_0\alpha_1\beta_{12} > 0$, $1 - \theta_1\pi_1 > 0$, $q_c - \beta\alpha_1\bar{a} > 0$, $(\beta_2 + C_0) - \bar{C} > 0$, $q - \alpha_0\bar{n} > 0$, $r_1(\beta_2 + C_0 - \bar{C}) - \pi_2\alpha_3\bar{a} > 0$

(iii) $E_3(n^*, a^*, C^*, P^*)$, Where

$$C^* = \frac{q_c - \beta\alpha_1 a^*}{\alpha_2},$$

$$a^* = \frac{r_1\alpha_2(\beta_2 + C_o) - r_1 q_c}{\pi_2\alpha_3\alpha_2 - \beta r_1\alpha_1},$$

$$n^* = \frac{[(q\beta_{11} + \pi_1\alpha_1 a^*\beta_{11}) - (\alpha_0\beta_{12} + \beta_1 a^*)] + \sqrt{[(q\beta_{11} + \pi_1\alpha_1 a^*\beta_{11}) - (\alpha_0\beta_{12} + \beta_1 a^*)]^2 + 4\alpha_0\beta_{11}(q\beta_{12} + \pi_1\alpha_1 a^*\beta_{12})}}{2\alpha_0\beta_{11}},$$

$$P^* = \frac{\theta_1\beta_1 n^*}{(\beta_{12} + \beta_{11}n^*)\alpha_3} - \frac{\alpha_1}{\alpha_3},$$

Thus, E_3 exist if

$q_c - \beta\alpha_1 a^* > 0$, $r_1\alpha_2(\beta_2 + C_0) - r_1 q_c > 0$, $\pi_2\alpha_3\alpha_2 - \beta r_1\alpha_1 > 0$

(4) DYNAMICAL BEHAVIOUR OF THE SYSTEM

In this section, we will discuss the stability behaviours of E_1, E_2 and E_3.

The variational matrix of model system (1) – (4) is given as follows:-

$$J_i = \begin{bmatrix} -\alpha_0 - \dfrac{a\beta_1\beta_{12}}{(\beta_{12} + \beta_{11}n)^2} & -\dfrac{\beta_1 n}{(\beta_{12} + \beta_{11}n)} + \pi_1\alpha_1 & 0 & 0 \\ \dfrac{\theta_1 a\beta_1\beta_{12}}{(\beta_{12} + \beta_{11}n)^2} & \dfrac{\theta_1\beta_1 n}{(\beta_{12} + \beta_{11}n)} - \alpha_1 - \alpha_3 P & 0 & -\alpha_3 a \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & \dfrac{\pi_2\alpha_3 P}{(\beta_2 + C_0 - C)} & \dfrac{\pi_2\alpha_3 aP}{(\beta_2 + C_o - C)^2} & \dfrac{\pi_2\alpha_3 a}{(\beta_2 + C_0 - C)} - r_1 \end{bmatrix}$$

Now, corresponding to the equilibrium point E_1, Jacobean J_1 is –

$$J_1 = \begin{bmatrix} -\alpha_0 & -\dfrac{\beta_1 q}{(\beta_{12}\alpha_0 + \beta_{11}q)} + \pi_1\alpha_1 & 0 & 0 \\ 0 & \dfrac{\theta_1\beta_1 q}{(\beta_{12}\alpha_0 + \beta_{11}q)} - \alpha_1 & 0 & 0 \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & 0 & 0 & -r_1 \end{bmatrix}$$

The Eigenvalues of the characteristic equation of J_1 are $\lambda_1 = -\alpha_0$, $\lambda_2 = -\alpha_2$,

$\lambda_3 = -r_1$ and $\lambda_4 = \dfrac{(\theta_1\beta_1 - \alpha_1\beta_{11})q - \alpha_1\alpha_0\beta_{12}}{(\beta_{12}\alpha_0 + \beta_{11}q)}$.

It is seen from the eigenvalues that the equilibrium E_1 is locally asymptotically stable if $(\beta_{11}q+\alpha_0\beta_{12})\alpha_1 > q\theta_1\beta_1$. Thus, E_2 is exist if E_1 is unstable.

Variation matrix corresponding to the equilibrium point E_2.

$$J_2=\begin{bmatrix} -\alpha_0 - \dfrac{\bar{a}\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}\bar{n})^2} & -\dfrac{\beta_1\bar{n}}{(\beta_{12}+\beta_{11}\bar{n})}+\pi_1\alpha_1 & 0 & 0 \\ \dfrac{\theta_1\bar{a}\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}\bar{n})^2} & \dfrac{\theta_1\beta_1\bar{n}}{(\beta_{12}+\beta_{11}\bar{n})}-\alpha_1 & 0 & -\alpha_3\bar{a} \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & 0 & 0 & \dfrac{\pi_2\alpha_3\bar{a}}{(\beta_2+C_0-\bar{C})}-r_1 \end{bmatrix}$$

using (1) – (4), above Jacobean converts to

$$J_2 = \begin{bmatrix} -a_{11} & \dfrac{(\alpha_0\bar{n}-q)}{\bar{a}} & 0 & 0 \\ a_{12} & 0 & 0 & -\alpha_3\bar{a} \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & 0 & 0 & \dfrac{\pi_2\alpha_3\bar{a}}{(\beta_2+C_0-\bar{C})}-r_1 \end{bmatrix}$$

Where $a_{11} = \alpha_0 + \dfrac{\bar{a}\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}\bar{n})^2}, a_{12} = \dfrac{\theta_1\bar{a}\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}\bar{n})^2}$

Characteristic equation corresponding to the above Jacobean is :-

$$(-\alpha_2-\lambda)\left(\frac{\pi_2\alpha_3\bar{a}}{(\beta_2+C_o-\bar{C})}-r_1-\lambda\right)\left[\lambda^2+a_{11}\lambda-\frac{(\alpha_0\bar{n}-q)}{\bar{a}}a_{12}\right]=0$$

The Eigenvalues of the characteristic equation of J_2 are $\lambda_1 = -\alpha_2, \lambda_2 = -\dfrac{\left[r_1\left((\beta_2+C_o)-\bar{C}\right)-\pi_2\alpha_3\bar{a}\right]}{\left[(\beta_2+C_o)-\bar{C}\right]}$,

$$\lambda = \frac{-a_{11} \pm \sqrt{{a_{11}}^2 + \dfrac{4(\alpha_0\bar{n}-q)}{\bar{a}}a_{12}}}{2}$$

$$\lambda_3 = -\left[\frac{a_{11} + \sqrt{{a_{11}}^2 + \dfrac{4(\alpha_0\bar{n}-q)}{\bar{a}}a_{12}}}{2}\right],$$

$$\lambda_4 = -\left[\frac{a_{11} - \sqrt{{a_{11}}^2 + \dfrac{4(\alpha_0\bar{n}-q)}{\bar{a}}a_{12}}}{2}\right],$$

Here λ_1, λ_2 , λ_3 and λ_4 are negative. Thus, point E_2 are stable .

Now, we will examine the local behavior of the equilibrium point E_3 (n^*, a^*, C^*, P^*). The Jacobean matrix corresponding to the equilibrium point E_3.

$$J_3=\begin{bmatrix} -\alpha_0-\frac{a^*\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}n^*)^2} & \frac{-\beta_1 n^*}{(\beta_{12}+\beta_{11}n^*)}+\pi_1\alpha_1 & 0 & 0 \\ \frac{a^*\theta_1\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}n^*)^2} & \frac{\theta_1\beta_1 n^*}{(\beta_{12}+\beta_{11}n^*)}-\alpha_1-\alpha_3 P^* & 0 & -\alpha_3 a^* \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & \frac{\pi_2\alpha_3 P^*}{(\beta_2+C_0-C^*)} & \frac{\pi_2\alpha_3 a^* P^*}{(\beta_2+C_0-C^*)^2} & \frac{\pi_2\alpha_3 a^*}{(\beta_2+C_0-C^*)}-r_1 \end{bmatrix}$$

using (1) – (4), above Jacobean converts to

$$J_3=\begin{bmatrix} -a_{11} & \frac{(\alpha_0 n^*-q)}{a^*} & 0 & 0 \\ a_{21} & 0 & 0 & -\alpha_3 a^* \\ 0 & -\beta\alpha_1 & -\alpha_2 & 0 \\ 0 & \frac{r_1 P^*}{a^*} & a_{43} & 0 \end{bmatrix}$$

Where $a_{11}=\alpha_0+\frac{a^*\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}n^*)^2}$, $a_{21}=\frac{a^*\theta_1\beta_1\beta_{12}}{(\beta_{12}+\beta_{11}n^*)^2}$, $a_{43}=\frac{\pi_2\alpha_3 a^* P^*}{(\beta_2+C_o-C^*)^2}$

Characteristic equation corresponding to the above Jacobean is -

$$\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0 \qquad (1)$$

where

$b_1=\alpha_2+a_{11}$

$b_2=a_{11}\alpha_2+r_1P^*\alpha_3+\frac{(q-\alpha_0 n^*)}{a^*}a_{21}$

$b_3=a_{11}r_1P^*\alpha_3+r_1P^*\alpha_2\alpha_3+\frac{(q-\alpha_0 n^*)}{a^*}\alpha_2 a_{21}-\beta a^*\alpha_1\alpha_3 a_{43}$

$b_4=\alpha_2\alpha_3 r_1 P^* a_{11}-\beta a^*\alpha_1\alpha_3 a_{43}a_{11}$

Using the Routh-hurwitz criteria, the condition for the stationary point to be locally asymptotically stable are $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $b_4 > 0$, $b_1b_2-b_3 > 0$ and $b_1b_2b_3 - b_3^2 - b_1^2 b_4 > 0$, we have shown it numerically using given set of parameters.

$q = 4$, $\beta_1 = 1$, $\alpha_0 = 1$, $\beta_{12} = 0.1$, $\beta_{11} = 1$, $\pi_1 = 0.9$, $\alpha_1 = 0.9$, $\theta_1 = 1$, $\alpha_3 = 0.1$,
$q_c = 10$, $\alpha_2 = 1$, $\beta = 0.9$, $\pi_2 = 0.1$, $\beta_2 = 0.3$, $C_0 = 25$, $r_1 = 0.001$, and
$a_{11} =1.0113677$, $a_{21} =0.0113677$, $a_{43} =0.0000444$, thus the values of b_1, b_2, b_3 and b_4 are as under $b_1 = 2.0113677$, $b_2 = 1.0133056$, $b_3 = 0.0020066$,
$b_4 = 0.0000686$ and $b_1b_2b_3 - b_3^2 - b_1^2 b_4 = 0.0038082 > 0$. All values are greater than zero, thus Jacobean matrix J_3 is asymptotically stable.

Now, In the following theorem we will study the nonlinear stability analysis of the equilibrium point E_3 by Lyapunovs direct method.

Theorem (1):- The equilibria E_3 is nonlinearly stable in Ω_1, if the following conditions are satisfied.

$$\left[\frac{(\beta_{12}\delta_m+\beta_{11}q)(\beta_{12}+\beta_{11}n^*)\pi_1\alpha_1-qn^*\beta_1\beta_{11}}{(\beta_{12}\delta_m+\beta_{11}q)(\beta_{12}+\beta_{11}n^*)}\right]^2<\frac{2\alpha_0 n^*\alpha_3}{3\theta_1} \qquad \text{.... (1.1)}$$

$$\beta^2\alpha_1^2 m_2 < \frac{n^*\alpha_3\alpha_2}{3\theta_1} \qquad \text{….(1.2)}$$

$$\left[\frac{m_3\pi_2\alpha_2}{(\beta_2\alpha_2 + C_0\alpha_2 - q_c)}\right]^2 < \frac{n^{*^2}}{6\theta_1^2} \qquad \text{. ….(1.3)}$$

$$\left[\frac{m_3\pi_2 a^*}{(\beta_2 + C_0 - C^*)(\beta_2\alpha_2 + C_0\alpha_2 - q_C)}\right]^2 \alpha_2\alpha_3 < \frac{m_2 n^*}{2\theta_1} \qquad \text{…….(1.4)}$$

Proof:-

We consider the following positive definite function:

$$V = \frac{1}{2}(n-n^*)^2 + m_1\left(a - a^* - a^* In\frac{a}{a^*}\right) + \frac{1}{2}m_2(C-C^*)^2 + m_3\left(P - P^* - P^* In\frac{P}{P^*}\right)$$

Where m_1, m_2 and m_3 are positive constants, to be chosen appropriately.

$$\frac{dV}{dt} = (n-n^*)\frac{dn}{dt} + m_1\frac{(a-a^*)}{a}\frac{da}{dt} + m_2(C-C^*)\frac{dC}{dt} + m_3\frac{(P-P^*)}{P}\frac{dP}{dt}$$

$$\frac{dV}{dt} = Z_1\frac{dn}{dt} + m_1\frac{Z_2}{a}\frac{da}{dt} + m_2 Z_3\frac{dC}{dt} + m_3\frac{Z_4}{P}\frac{dP}{dt}$$

We assume $Z_1 = (n - n^*)$, $Z_2 = (a - a^*)$, $Z_3 = (C - C^*)$, $Z_4 = (P - P^*)$

Using (1) – (4), choosing $m_1 = \frac{n^*}{\theta_1}$ and using the inequality $a^2 + b^2 \geq 2ab$, then some algebraic manipulations $\frac{dV}{dt}$ reduces in the following form:

$$\begin{aligned}\frac{dV}{dt} \leq & -\frac{\beta_1\beta_{12}a}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}Z_1^2 \\
& -\frac{1}{2}2\alpha_0 Z_1^2 + \left[\pi_1\alpha_1 - \frac{nn^*\beta_1\beta_{11}}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}\right]Z_1Z_2 - \frac{1}{2}\frac{n^*\alpha_3}{3\theta_1}Z_2^2 \\
& -\frac{1}{2}\frac{n^*\alpha_3}{3\theta_1}Z_2^2 - m_2\beta\alpha_1 Z_2Z_3 - \frac{1}{2}m_2\alpha_2 Z_3^2 \\
& -\frac{1}{2}\frac{n^*\alpha_3}{3\theta_1}Z_2^2 + \frac{m_3\pi_2\alpha_3}{(\beta_2+C_0-C)}Z_2Z_4 - \frac{1}{2}\frac{n^*\alpha_3}{2\theta_1}Z_4^2 \\
& -\frac{1}{2}m_2\alpha_2 Z_3^2 + \frac{m_3\pi_2\alpha_3 a^*}{(\beta_2+C_0-C)(\beta_2+C_0-C^*)}Z_3Z_4 - \frac{1}{2}\frac{n^*\alpha_3}{2\theta_1}Z_4^2\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt} \leq & -\frac{\beta_1\beta_{12}a}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}Z_1^2 \\
& -\frac{1}{2}P_{11}Z_1^2 + P_{12}Z_1Z_2 - \frac{1}{2}P_{22}Z_2^2 \\
& -\frac{1}{2}P_{22}Z_2^2 + P_{23}Z_2Z_3 - \frac{1}{2}P_{33}Z_3^2 \\
& -\frac{1}{2}P_{22}Z_2^2 + P_{24}Z_2Z_4 - \frac{1}{2}P_{44}Z_4^2 \\
& -\frac{1}{2}P_{33}Z_3^2 + P_{34}Z_3Z_4 - \frac{1}{2}P_{44}Z_4^2\end{aligned}$$

Where,

$$P_{11} = 2\alpha_0, P_{12} = \left[\pi_1\alpha_1 - \frac{nn^*\beta_1\beta_{11}}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}\right], P_{22} = \frac{n^*\alpha_3}{3\theta_1},$$

$$P_{23} = -m_2\beta\alpha_1, P_{33} = m_2\alpha_2, P_{24} = \frac{m_3\pi_2\alpha_3}{(\beta_2+C_0-C)}, P_{34} = \frac{m_3\pi_2\alpha_3 a^*}{(\beta_2+C_0-C)(\beta_2+C_0-C^*)},$$

$$P_{44} = \frac{n^*\alpha_3}{2\theta_1}$$

Thus, sufficient conditions for $\frac{dV}{dt}$ to be negative definite in Ω_1 are that the following inequalities hold:

$$P_{12}^2 < P_{11}.P_{22}, P_{23}^2 < P_{22}.P_{33}, P_{24}^2 < P_{22}.P_{44}, P_{34}^2 < P_{33}.P_{44}.$$

Hence, V is a Lyapunov's function with respect to E_3 whose domain contains the region of attraction Ω_1, proving the theorem.

(5) NUMERICAL SIMULATION

To check the feasibility of our analysis regarding the existence of E_3 and the corresponding stability conditions, we conduct with the numerical computation of model (1) – (4) by choosing the following values of the parameters:

$q = 4$, $\beta_1 = 1$, $\alpha_0 = 1$, $\beta_{12} = 0.1$, $\beta_{11} = 1$, $\pi_1 = 0.9$,
$\alpha_1 = 0.9$, $\theta_1 = 1$, $\alpha_3 = 0.1$, $q_c = 10$, $\alpha_2 = 1$, $\beta = 0.9$,
$\pi_2 = 0.1$, $\beta_2 = 0.3$, $C_0 = 25$, $r_1 = 0.001$.

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_3(n^*, a^*, C^*, P^*)$ are satisfied and E_3 is given by

$n^* = 3.7270$, $a^* = 1.6649$, $C^* = 8.6515$, $P^* = $ o.7388.

It is pointed out here that for the above set of parameters, the conditions for nonlinear stability (1.1), (1.2), (1.3) and (1.4) are also satisfied .

In figure (1), we observed that the equilibrium point E_3 is asymptotically stable. In figure, concentration of dissolved oxygen and nutrients increases, while algae and zooplankton population decreases. It is further noted that all the stability conditions satisfied for the above values of parameters showing the local and nonlinear stability behaviour of E_3.

Time Series Graph

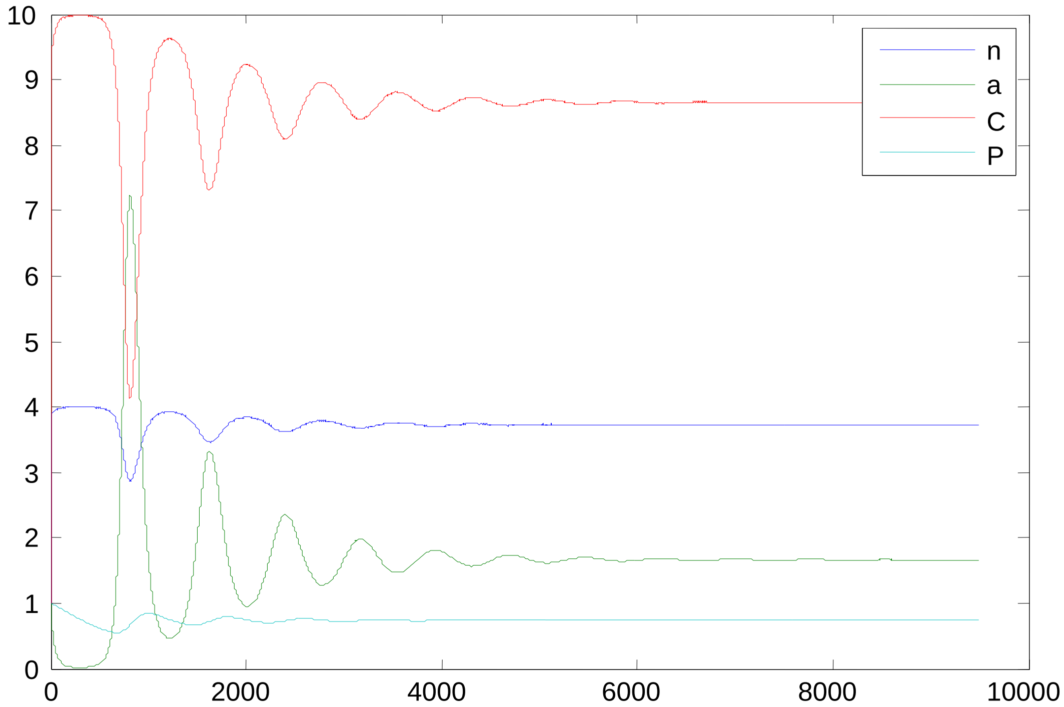

Fig.-1

(6) CONCLUSION

In this chapter, a non-linear mathematical model for the depletion of dissolved of oxygen in plankton ecosystem has been proposed and analyzed. The model has three feasible steady states (equilibria) E_1, E_2 and E_3.

After analyzing the stability of equilibrium points we have seen that all the feasible equilibria have locally asymptotically stable under the certain conditions. We have studied the nonlinear stability analysis of the interior equilibrium point E_3 by Lyapunovs direct method.

By numerical simulation it is shown that concentration of dissolved oxygen and nutrients increases, while density of algae and zooplankton population decreases. Numerical example is considered to show the stability with the help of figure (1), which support the qualitative analysis too.

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