Certain Transformation Formulae for Poly-Basic Hypergeometric Series

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Abstract : Making use of Bailey's transformation and certain known summations of truncated series, an attempt has been made to establish transformation formulae involving polybasic hypergeometric series. We offer an overview of some of the main findings from the hypergeometric sequence theories and integrals associated with root systems. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric using some known summation formulae and the identity defined herein. In particular, for such multiple series and integrals, we list a number of summations, transformations and explicit evaluations. Interesting transformation formulas for poly - basic hypergeometric sequence have been constructed using some known summation formulae and the identity set out herein.
Keyword : Polybasic, hypergeometric, transformation, Hypergeometric Series.

Introduction

The hypergeometric function and its generalizations, summation theorems and transformation formulae have been presented in many textbooks. Mathematicians working in the area of ordinary and basic hypergeometric series were interested for transformation formulae among various generalised hypergeometric functions and they succeeded in their goal. The celebrated Bailey [1] transform was extensively used to obtain transformation formulae of ordinary hypergeometric series and basic hypergeometric series with help of known summation formulae.

In view of the importance and usefulness of the generating relations, we have extended the idea of generating relations for obtaining transformation formulae of ordinary hypergeometric series. The transformation formulae of hypergeometric series play a pivotal role in the investigation of various useful properties and can also be used as a new platform for further study.[2]

In 1944, Bailey [3] established a powerful series identity which was later known as Bailey's lemma. The Bailey's lemma states that, if

$$\beta_n = \sum_{r=0}^{n} \alpha_r \, u_{n-r} \, u_{n+r}$$

And

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \, u_{r-n} \, u_{n+r} \qquad (1.1)$$

then, under the suitable convergence conditions and if change in the order of summations is allowable [4]

$$\sum_{n=0}^{\infty} \alpha_r \, \gamma_n = \sum_{n=0}^{\infty} \beta_n \, \delta_n$$

where α_r, δ_r, u_r and υ_r are functions of r, such that γ_n exists. The proof of the lemma is trivial [5]

The following notations and definitions shall be used throughout this paper For 'a' real or complex and 'n' be a positive integer, we define [6]

$$(a)_0 = 1$$
$$(a)_n = a(a+1)(a+2)\ldots(a+n-1), \qquad n = 1,2,3, \ldots.$$

If 'a' is a negative integer -m , then

$$(a)_n = (-m)_n \quad if\ m \geq n$$
$$(a)_0 = 0 \quad if\ m < n$$

Now, we define a generalized hypergeometric function,

$$rFs\left[\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s; \end{matrix} z\right] = rFs\left[\begin{matrix} (a_r); \\ (b_s); \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n \, z^n}{(b_1)_n (b_2)_n \ldots (b_s)_n \, (1)_n} = \sum_{n=0}^{\infty} \frac{((a_r))_n \, z^n}{((b_s))_n \, (1)_n} \qquad (1)$$

Where there are always r of a parameters and s of the b parameters. The meaning of (a) and (b) are sequences of parameters a_1,a_2,a_3.......a_r and b_1,b2,b3....bs respectively.[7]

The series (1) is convergent if

i) $Rl[\sum_{v=1}^{s} b_v - \sum_{v=1}^{r} a_v] > 0 \quad when\ z = 1$
ii) $Rl[\sum_{v=1}^{s} b_v - \sum_{v=1}^{r} a_v] > -1 \quad when\ z = -1$
iii) $r = s + 1 \quad when\ |z| < 1$
iv) $r > s + 1 \quad when\ z = 0$

A Basic Hypergeometric Series is defined as

$$r\Phi s\begin{bmatrix} a_1, a_2, \dots, a_r; \\ b_1, b_2, \dots, b_s; \end{bmatrix} q; z\Big] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n (b_2;q)_n \dots (b_s;q)_n} \left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} z^n$$

For 0 < |q| < 1, the series converges absolutely for all z if r ≤ s and for |z| < 1 if r = s + 1
This series also converges absolutely if |q| > 1 and |z| < |b1b2...bs|/|a1a2...ar|.
We define the Poly-Basic Hypergeometric Series as[8]

$$\Phi\begin{bmatrix} a_1, a_2, \dots, a_r & : c_{1,1}, \dots, c_{1,r_1} & ; \dots ; c_{m,1}; \dots, c_{m,rm} & ; q, q_1 q_2 \dots q_m \\ b_1, b_2, \dots, b_s & : d_{1,1}, \dots, d_{1,s_1} & ; \dots ; d_{m,1}; \dots, d_{m,sm} & \end{bmatrix} = \sum_{t=0}^{n-1} \frac{(a_1, a_2, \dots, a_r; q)_t}{(q, b_1, b_2, \dots, b_s; q)_t} z^t \prod_{j=1}^{m} \frac{(c_{j,1} \dots c_{j,r_j}; q_j)_t}{(d_{j,1} \dots d_{j,s_j}; q_j)_t}$$

which converges for max (|q|, |q1|, ..., |qm|) < 1
We shall also require the following known results in our work

$$2\Phi1\begin{bmatrix} \alpha, & t, & q; & q \\ & ayq & & \end{bmatrix}_n = \frac{(\alpha q, yq; q)_n}{(q, ayq; q)_n}$$

$$4\Phi3\begin{bmatrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & e; & q; 1/e \\ & \sqrt{a}, & -\sqrt{\alpha}, & \alpha q / e & \end{bmatrix}_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q / e; q)_n e^n}$$

$$6\Phi5\begin{bmatrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & \beta, & \gamma, & \delta; & q; & q \\ & \sqrt{a}, & -\sqrt{\alpha}, & \alpha q/\beta, & \alpha q/\gamma, & \alpha q / \delta & & \end{bmatrix}_n = \frac{(\alpha q, \beta q, \gamma q, \delta q; q)_n}{(q, \alpha q/\beta, \alpha q/\gamma, \alpha q / \delta; q)_n}$$

where α = βγδ

$$\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(\alpha;p)_i (\beta;q)_i \beta^{-i}}{(1-\alpha)(q;q)_i (\alpha p / \beta; p)_i} = \frac{(\alpha p;p)_n (\beta q;q)_n \beta^{-n}}{(q;q)_n (\alpha p / \beta; p)_i}$$

$$\sum_{i=0}^{n} \frac{(1-\alpha p^i q^i)(1-\beta p^i q^{-i})(\alpha;p)_i (\beta;p)_i (\gamma;q)_i (\alpha / \beta\gamma; q)_i q^i}{(1-\alpha)(1-\beta)(q;q)_i (\alpha q / \beta; q)_i (\alpha p / \gamma; p)_i (\beta\gamma p; p)_i} = \frac{(\alpha p, \beta p; p)_n (\gamma q; q)_n (\alpha q / \beta\gamma; q)_n}{(q, \alpha q / \beta; q)_n (\alpha p / \gamma; p)_n (\beta\gamma p; p)_n}$$

$$\sum_{r=0}^{n} \frac{(1-\alpha\delta p^r q^r)(1-\beta p^r / \delta q^r)(\alpha, \beta; p)_r (\gamma, \alpha\delta^2 / \beta\gamma; q)_r}{(1-\alpha\delta)(1-\beta / \delta)(\delta q, \alpha\delta q / \beta; q)_r (\alpha\delta p/\gamma, \beta\gamma p / \delta; p)_r} q^r$$

$$\frac{(1-\alpha)(1-\beta)(1-\gamma)(1-\alpha\delta^2 / \beta\gamma)}{\delta(1-\alpha\delta)(1-\beta / \delta)(1-\gamma / \delta)(1-\alpha\delta / \beta\gamma)}$$

$$\times\left(\frac{(\alpha p, \beta p; p)_n (\gamma q, \alpha\delta^2 q/\beta\gamma; q)_n}{(\delta q, \alpha\delta q/\beta; q)_n (\alpha\delta p / y, \beta\gamma p/\delta; p)_n} - \frac{(\gamma / \alpha\delta, \delta/\beta\gamma; p)_1 (1 / \delta, \beta/\alpha\delta; q)_1}{(1 / \gamma, \beta\gamma/\alpha\delta^2; q)_1 (1 / \alpha, 1/\beta; p)_1}\right)$$

Research Methodology

Research Methodology refers the discussion regarding the specific methods chosen and used in a research paper. This discussion also encompasses the theoretical concepts that further provide information about the methods selection and application. The current study is descriptive in nature and is based on secondary data gathered from a variety of sources, including books , education, and development, journals, scholarly articles, government publications, and printed and online reference materials.

Result and Discussion

In this section we have established the following main results[9-13]

$$\Phi\begin{bmatrix} \alpha q, \beta q: a, y; \\ \alpha\beta q: p, ayp; \end{bmatrix} q.p;p\Big] = \frac{[ap, yp; p]_\infty \ [\alpha q, \beta q; q]_\infty}{[p, ayp; p]_\infty \ [q, \alpha\beta q; q]_\infty} - \frac{q(1-\alpha)(1-\beta)}{(1-q)(1-\alpha\beta q)} \Phi\begin{bmatrix} ap, yp: \alpha q, \beta q; \\ ayp: q^2, \alpha\beta q^2; \end{bmatrix} p, q; q\Big], \tag{1.2}$$

$$\Phi\begin{bmatrix} \alpha q, eq: a, y; \\ \frac{\alpha q}{e}: p, ayp; \end{bmatrix} q.p;\frac{p}{e}\Big] = \frac{(1-\alpha q^2)(1-e)}{e(1-q)(1-\alpha q / e)} \times \Phi\begin{bmatrix} ap, yp: \alpha q, q^2\sqrt{\alpha}, -q^2\sqrt{\alpha}, eq; \\ ayp: q^2 q\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\alpha q^2}{e}; \end{bmatrix} p, q; \frac{1}{e}\Big],$$

$$\Phi\begin{bmatrix} \alpha q, \beta q: \gamma q, \delta q: a, y; \\ \frac{\alpha q}{\beta}, \frac{\alpha q}{\gamma}, \frac{\alpha q}{\delta}: p, ayp; \end{bmatrix} q.p;p\Big]$$

(1.3)

$$= \frac{[ap, yp; p]_\infty \ [\alpha q, \beta q, \gamma q, \delta q; q]_\infty}{[p, ayp; p]_\infty \ [q, \alpha q/\beta, \gamma, \alpha q/\delta; q]_\infty} - \frac{q(1-q^2\alpha)(1-\beta)(1-\gamma)(1-\delta)q}{(1-q)(1-\alpha q/\beta)(1-\alpha q/\gamma)(1-\alpha q/\delta)}$$

$$\times \Phi\begin{bmatrix} ap, yp: \alpha q, q^2\sqrt{\alpha}, -q^2\sqrt{\alpha}, \beta q, \gamma q, \delta q; \\ ayp: q^2 q\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\alpha q^2}{\beta}, \frac{\alpha q^2}{\gamma}, \frac{\alpha q^2}{\delta}; \end{bmatrix} p, q; q\Big],$$

$$\Phi\begin{bmatrix} x, y: ap: cp; \\ xyP: \frac{\alpha p}{c}: q; \end{bmatrix} P, p, q; \frac{p}{c}\Big] = \frac{(1-apq)(1-c)}{(1-q)(1-ap/c)c} \times \Phi\begin{bmatrix} xP, yP: ap: cq: ap^2q^2; \\ xyP: \frac{ap^2}{c}: q^2: apq; \end{bmatrix} P, p, q, pq; \frac{1}{c}\Big],$$

$$\Phi\left[\begin{matrix} x,y:ap:bq:cq,\frac{aq}{bc}; \\ xyP:\frac{ap}{c}:bcp:q,\frac{aq}{b}; \end{matrix} P,p,q,P\right]$$

$$= \frac{[xP,yP;P]_\infty\ [ap,bp;p]_\infty[cq,aq\ /\ bc;q]_\infty}{[P,xyP;P]_\infty\ [q,aq/b\,;q]_\infty[ap\ /\ c,bcp;p]_\infty} - \frac{(1-apq)(1-bp/q)(1-c)(1-a/bc)}{(1-q)(1-aq/b)(1-ap/c)(1-bcp)} \tag{1.4}$$

$$\times\,\Phi\left[\begin{matrix} xP,yP:ap^2q^2:\frac{bp^2}{q^2}:ap,bp:cq,\frac{aq}{bc}; \\ xyP:apq:\frac{bp}{q}:\frac{ap^2}{c},bcp^2:q^2,\frac{aq^2}{b}; \end{matrix} P,pq,\frac{p}{q},p,q:q\right],$$

$$\Phi\left[\begin{matrix} x,y:ap:bp:cq,\frac{ad^2q}{bc}; \\ xyP:\frac{adp}{c}:\frac{bcp}{d}:dq,\frac{adq}{b}; \end{matrix} P,p,q,P\right]$$

$$= \frac{[xP,yP;P]_\infty\ [ap,bp;p]_\infty[cq,ad^2q\ /\ bc;q]_\infty}{[P,xyP;P]_\infty\ [dq,adq/b\,;q]_\infty[adp\ /\ c,bcp\ /\ d;p]_\infty} - \frac{dq(1-adpq)(1-bp/dq)(1-c/d)(1-ad/bc)}{(1-dq)(1-adq/b)(1-adp/c)(1-bcp/d)} \tag{1.5}$$

$$\times\,\Phi\left[\begin{matrix} xP,yP:adp^2q^2:\frac{bp^2}{dq^2}:ap,bp:cq,\frac{ad^2q}{bc}; \\ xyP:adpq:\frac{bp}{dq}:\frac{adp^2}{c},\frac{bcp^2}{d}:dq^2,\frac{adq^2}{b}; \end{matrix} P,pq,\frac{p}{q},p,q:q\right] \tag{1.6}$$

$$= \frac{[xP,yP;P]_\infty\ [ap,bp;p]_\infty[cq,ad^2q\ /\ bc;q]_\infty}{[P,xyP;P]_\infty\ [dq,adq/b\,;q]_\infty[adp\ /\ c,bcp\ /\ d;p]_\infty}$$

$$- \frac{dq(1-adpq)(1-bp/dq)(1-c/d)(1-ad/bc)}{(1-dq)(1-adq/b)(1-adp/c)(1-bcp/d)}$$

$$\times\,\Phi\left[\begin{matrix} xP,yP:adp^2q^2:\frac{bp^2}{dq^2}:ap,bp:cq,\frac{ad^2q}{bc}; \\ xyP:adpq:\frac{bp}{dq}:\frac{adp^2}{c},\frac{bcp^2}{d}:dq^2,\frac{adq^2}{b}; \end{matrix} P,pq,\frac{p}{q},p,q:q\right]$$

Proof of Main Results

Taking u_r=v_r-1 in 1.1, Bailey's transformation takes the following form [14]

$$if\ \beta_n = \sum_{r=0}^{n} \alpha_r, \tag{1.7}$$

$$\gamma_n = \sum_{r=0}^{\infty} \delta_r \tag{1.8}$$

$$then\ \sum_{n=0}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty} \beta_r\delta_r. \tag{1.9}$$

Proof of Result 1.2.

Taking, $\alpha_r = (\alpha,\beta;q)_r q^r\ /\ (q,\alpha\beta q;q)_r$ and respectively, and $\delta_r = (a,y;q)_r p^r\ /\ (p,ayp;p)_r$ making use of 2.7, we get in 1.7 , 1.8 , respectively, we get

$$\beta_n = \frac{(\alpha q,\beta q;q)_n}{(q,\alpha\beta q;q)_n}, \qquad \gamma_n = \frac{(ap,yp;p)_\infty}{(p,ayp;p)_\infty} - \frac{(1-ay)(1-p^n)(a,y;p)_n}{(1-a)(1-y)(p,ay;p)_n}$$

Putting these values in 1,9, we get the following transformation

$$\Phi\left[\begin{matrix}\alpha q,\beta q:a,y; \\ \alpha\beta q:p,ayp;\end{matrix} q,p;p\right] + \frac{(1-ay)}{(a-1)(1-y)}\Phi\left[\begin{matrix}\alpha,\beta:a,y; \\ \alpha\beta q:p,ay;\end{matrix} q,p;q\right]$$

$$= \frac{(ap,yp;p)_\infty}{(p,ayp;p)_\infty}\frac{(\alpha q,\beta q;q)_\infty}{(q,\alpha\beta q;q)_\infty} + \frac{(1-ay)}{(1-a)(1-y)}\Phi\left[\begin{matrix}\alpha,\beta:a,y; \\ \alpha\beta q:p,ay;\end{matrix} q,p;pq\right]$$

which on simplification gives the result 1.2

Proof of Result 1.3

Taking

$$\alpha_r = \left(\alpha,q\sqrt{\alpha},-q\sqrt{\alpha},e;q\right)_r\ /\ (q,\sqrt{\alpha},-\sqrt{\alpha},\alpha q/e\,;q)_r e^r$$

And $\delta_r = (a, y; p)_r p^r \,/\, (p, ayp; p)_r$ in 1.7,1.8 respectively, we get

$$\beta_n = \frac{(\alpha q, eq; q)_n}{(q, \alpha q/e\,; q)_n e^n}, \qquad \gamma_n = \frac{(ap, yp; p)_\infty}{(p, ayp; p)_\infty} - \frac{(1-ay)(1-p^n)(a, y; p)_n}{(1-a)(1-y)(p, ay; p)_n}$$

Substituting these values in 1.9, we get the following transformation for |e| > 1:[15]

$$\Phi\left[\begin{matrix}\alpha q, eq: a, y; \\ \frac{\alpha q}{e}: p, ayp;\end{matrix}\; q, p; \frac{p}{e}\right] = \frac{(1-ay)}{(1-a)(1-e)} \times \Phi\left[\begin{matrix}\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{e}: p, ay;\end{matrix}\; q, p; \frac{p}{e}\right]$$

$$-\frac{(1-ay)}{(1-a)(1-y)}\Phi\left[\begin{matrix}\alpha, q\sqrt{\alpha}, -q\sqrt{\alpha}, e: a, y; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{\alpha q}{e}: p, ay;\end{matrix}\; q, p; \frac{1}{e}\right],$$

Conclusion

In the above section, we have demonstrated the power of Bailey lemma as a tool for discovering new transformations of basic hypergeometric series from the known summations and transformations. Some of the transformations in the previous section generalize the known transformation formulae.

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