Characterization and Estimation of Length Biased Two Parameter Rani Distribution with Applications

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Abstract: In this paper, we have executed a new distribution called as length biased two parameter Rani distribution by using the length biased technique. Various statistical properties of newly proposed distribution including moments, survival function, hazard rate function, reverse hazard rate function, order statistics, entropies, bonferroni and lorenz curves have been derived and discussed. Estimation of parameters has also been discussed by using the technique of maximum likelihood estimation and its Fisher's information matrix has been obtained. Finally, a real life data set has been used to examine the ability and superiority of newly proposed distribution.

Keywords: Two parameter Rani distribution, Weighted distribution, Survival analysis, Order statistics, Maximum likelihood estimation.

1. Introduction

Fisher (1934) introduced the concept of weighted distribution is a traceable work in respect of his studies on how the methods of ascertainment can affect the form of distribution of recorded observations. Later it was introduced and formulated in a more general way by Rao (1965) with respect to modeling statistical data where the routine practice of using standard distributions for the purpose was dismissed as inappropriate. The study of weighted distributions are useful in distribution theory because it provides a new understanding of the existing standard probability distributions and it provides methods for extending existing standard probability distributions for modeling lifetime data due to introduction of additional parameter in the model which creates flexibility in their nature. Weighted distributions occur in modeling clustered sampling, heterogeneity and extraneous variation in the data set. The theory of weighted distributions provides a collective access for the problem of model specification and data interpretation. The weighted distributions are widely used in many fields such as medicine, ecology, reliability, meta analysis, analysis of family data, analysis of intervention data and other areas for the improvement of proper statistical models. The weighted distributions are utilised to modulate the probabilities of events as observed and transcribed. The weighted distributions are used as a tool in selection of appropriate models for observed data, especially when samples are drawn without a proper frame. The weighted distributions take into account the method of ascertainment by adjusting the probabilities of the actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Patil and Rao (1978) introduced the concept of size biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. Warren (1975) was the first to apply the weighted distributions in connection with sampling wood cells. The weighted distribution reduces to length biased distribution when the weight function considers only the length of units of interest. The concept of length biased sampling was introduced by Cox (1969) and Zelen (1974). More generally, when the sampling mechanism selects units with probability proportional to measure of the unit size, resulting distribution is called size biased. Size biased distributions are a special case of weighted distributions. A lot of work has been done by many researchers to develop some important length biased probability models with their significant role in handling data sets from various practical fields. Moniem and Diab (2018) discussed on the length-biased weighted exponentiated Lomax distribution. Seenoi et al. (2014) introduced the length biased exponentiated inverted Weibull distribution. Shaban and Boudrissa (2007) studied the weibull length biased distribution with properties and estimation. Vijayakumar et al. (2020) presented the length biased Rani distribution with survival data analysis. Karimi and Nasiri (2018) studied the length biased weighted Lomax distribution in the presence of outliers. Elfattah et al. (2021) discussed on the length biased Burr-XII distribution with properties and applications. Saghir et al. (2016) studied the length-biased weighted exponentiated inverted weibull distribution. Ganaie and Rajagopalan (2021) presented the length biased power quasi lindley distribution with applications on real lifetime data. Gupta and Tripathi (1996) studied the weighted version of the bivariate three parameter logarithmic series distribution which has application in many fields such as ecology, social and behavioral sciences and species abundance studies. Aleem et al. (2013) introduced a class of modified weighted weibull distribution (MWWD) and its properties. Recently, Mustafa and Khan (2022) studied the length biased powered inverse Rayleigh distribution with applications.

The two parameter Rani distribution proposed by Al-omari et al. (2021) is a newly introduced life time distribution, which is a special case of one parameter Rani distribution. Its various structural properties including its moments, moment generating function, reliability function, hazard rate function, reverse hazard rate function, odds function, density function, order statistics, stochastic ordering and entropies have been discussed. The estimation of parameters has also been discussed through the method of maximum likelihood estimation. Shanker (2017) studied Rani distribution and its application, discuss its various mathematical and statistical properties and estimate its parameters by method of moments and method of maximum likelihood estimation. An application has been presented from fitting aluminum reduction cells sets and strength data of glass of aircraft window to investigate the superiority of suggested two parameter Rani distribution.

2. Length Biased Two Parameter Rani (LBTPR) Distribution

The probability density function of two parameter Rani (TPR) distribution is given by

$$f(x;\theta,\alpha)=\frac{\theta^5}{\alpha\theta^5+24}\left(\alpha\theta+x^4\right)e^{-\theta x};\ x>0,\theta>0,\alpha>0 \qquad (1)$$

and the cumulative distribution function of two parameter Rani distribution is given by

$$F(x;\theta,\alpha)=1-\left(1+\frac{24\theta x+\theta^2x^2[12+x\theta(4+x\theta)]}{24+\alpha\theta^5}\right)e^{-\theta x};x>0,\theta>0,\alpha>0 \qquad (2)$$

Let X be the random variable following non-negative condition has probability density function $f(x)$. Let $w(x)$ be its non-negative weight function, then the probability density function of weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Where $w(x)$ be the non - negative weight function and $E(w(x))=\int w(x)f(x)dx<\infty$.

In this paper, we have to obtain the length biased version of two parameter Rani distribution, we have considered the weight function as $w(x) = x$ to obtain the length biased two parameter Rani distribution. Then, the probability density function of length biased distribution is given by

$$f_l(x)=\frac{x\,f(x)}{E(x)} \qquad (3)$$

Where $E(x)=\int_0^\infty x\,f(x;\theta,\alpha)dx$

$$E(x)=\frac{\alpha\theta^5+120}{\theta(\alpha\theta^5+24)} \qquad (4)$$

Substitute equations (1) and (4) in equation (3), we will obtain the probability density function of length biased two parameter Rani distribution as

$$f_l(x)=\frac{\theta^6}{\alpha\theta^5+120}x(\alpha\theta+x^4)e^{-\theta x} \qquad (5)$$

and the cumulative distribution function of length biased two parameter Rani distribution can be obtained as

$$F_l(x)=\int_0^x f_l(x)dx$$

$$F_l(x)=\int_0^x\frac{\theta^6}{\alpha\,\theta^5+120}x\left(\alpha\theta+x^4\right)e^{-\theta x}dx$$

$$F_l(x)=\frac{1}{\alpha\,\theta^5+120}\int_0^x\theta^6x\left(\alpha\theta+x^4\right)e^{-\theta x}dx$$

$$F_l(x)=\frac{1}{\alpha\theta^5+120}\left(\alpha\theta^7\int_0^x x\,e^{-\theta x}dx+\theta^6\int_0^x x^5e^{-\theta x}dx\right) \qquad (6)$$

Put $\theta x=t\quad\Rightarrow\theta dx=dt\quad\Rightarrow dx=\frac{dt}{\theta}$, when $x\to x,t\to\theta x,\ x\to0,t\to0$

Also $x=\frac{t}{\theta}$

After simplification of equation (6), we will obtain the cumulative distribution function of length biased two parameter Rani distribution as

$$F_l(x)=\frac{1}{\alpha\theta^5+120}\left(\alpha\theta^5\gamma(2,\ \theta x)+\gamma(6,\ \theta x)\right) \qquad (7)$$

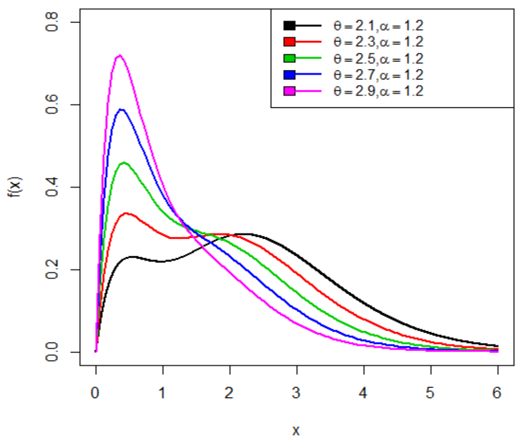

Fig.1:Pdf plot of Length Biased Two Parameter Rani Distribution

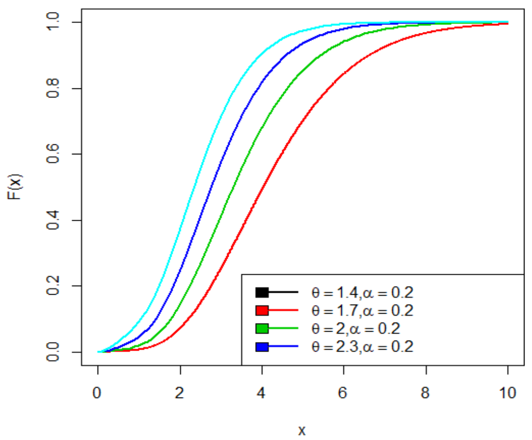

Fig.2:Cdf plot of Length Biased Two Parameter Rani Distribution

3. Survival Analysis

In this section, we will obtain the survival function, hazard rate and reverse hazard rate functions of the length biased two parameter Rani distribution.

a. Survival function

The survival function is also called reliability function or compliment of the cumulative distribution function and the survival function can be obtained as

$$S(x) = 1 - F_l(x)$$

$$S(x) = 1 - \frac{1}{\alpha\theta^5 + 120}\left(\alpha\theta^5 \gamma(2, \theta x) + \gamma(6, \theta x)\right)$$

b. Hazard function

The hazard function is also known as hazard rate or failure rate or force of mortality and is given by

$$h(x) = \frac{f_l(x)}{1 - F_l(x)}$$

$$h(x) = \frac{x\theta^6(\alpha\theta + x^4)e^{-\theta x}}{(\alpha\theta^5 + 120) - (\alpha\theta^5 \gamma(2, \theta x) + \gamma(6, \theta x))}$$

c. Reverse hazard function

The reverse hazard function is given by

$$h_r(x) = \frac{f_l(x)}{F_l(x)}$$

$$h_r(x) = \frac{x\theta^6(\alpha\theta + x^4)e^{-\theta x}}{(\alpha\theta^5 \gamma(2, \theta x) + \gamma(6, \theta x))}$$

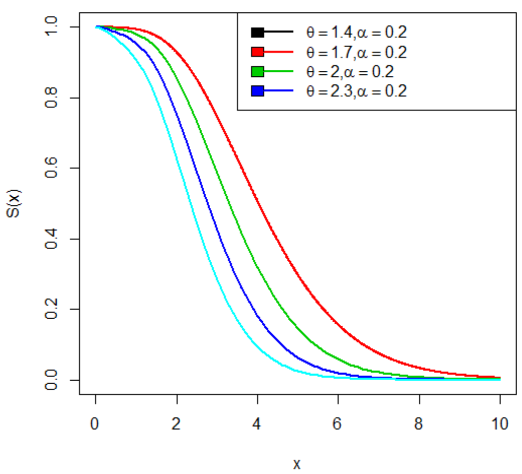

Fig.3:Survival plot of Length Biased Two Parameter Rani Distribution

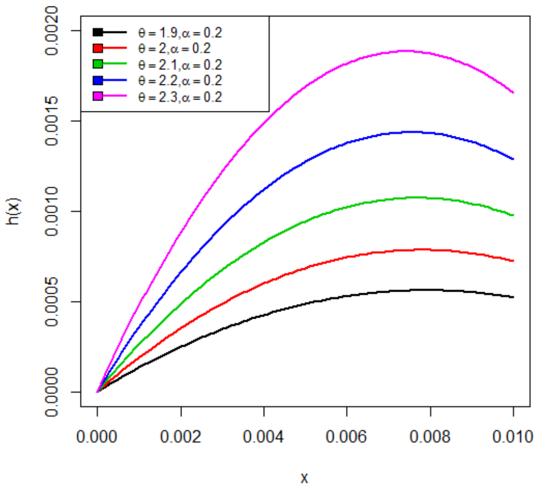

Fig.4:Hazard Plot of Length Biased Two Parameter Rani Distribution

4. Structural properties

In this section, we have obtained various statistical properties of length biased two parameter Rani distribution including its moments, harmonic mean, MGF and characteristic function.

4.1 Moments

Let X be the random variable following length biased two parameter Rani distribution with parameters θ and α, then the r^{th} order moment $E(X^r)$ of X about origin can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_l(x)dx$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r \frac{\theta^6}{\alpha\theta^5+120} x(\alpha\theta + x^4)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \int_0^\infty \frac{\theta^6}{\alpha\theta^5+120} x^{r+1}(\alpha\theta + x^4)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \frac{\theta^6}{\alpha\theta^5+120} \int_0^\infty x^{r+1}(\alpha\theta + x^4)e^{-\theta x}dx$$

$$E(X^r) = \mu_r' = \frac{\theta^6}{\alpha\theta^5+120}\left(\alpha\theta\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx + \int_0^\infty x^{(r+6)-1}e^{-\theta x}dx\right) \tag{8}$$

After simplification, equation (8) becomes

$$E(X^r) = \mu_r' = \frac{\alpha\theta^5\Gamma(r+2)+\Gamma(r+6)}{\theta^r(\alpha\theta^5+120)} \tag{9}$$

Putting r = 1, 2, 3 and 4 in equation (9), we will obtain first four moments of length biased two parameter Rani distribution as.

$$E(X) = \mu_1' = \frac{2\alpha\theta^5+720}{\theta(\alpha\theta^5+120)}$$

$$E(X^2) = \mu_2' = \frac{6\alpha\theta^5+5040}{\theta^2(\alpha\theta^5+120)}$$

$$E(X^3) = \mu_3' = \frac{24\alpha\theta^5+40320}{\theta^3(\alpha\theta^5+120)}$$

$$E(X^4) = \mu_4' = \frac{120\alpha\theta^5+362880}{\theta^4(\alpha\theta^5+120)}$$

$$\text{Variance} = \frac{6\alpha\theta^5+5040}{\theta^2(\alpha\theta^5+120)} - \left(\frac{2\alpha\theta^5+720}{\theta(\alpha\theta^5+120)}\right)^2$$

$$S.D(\sigma) = \sqrt{\frac{6\alpha\theta^5+5040}{\theta^2(\alpha\theta^5+120)} - \left(\frac{2\alpha\theta^5+720}{\theta(\alpha\theta^5+120)}\right)^2}$$

4.2 Harmonic mean

The harmonic mean for the proposed model can be obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x)dx$$

$$H.M = \int_0^\infty \frac{\theta^6}{\alpha\theta^5+120}(\alpha\theta + x^4)e^{-\theta x}dx$$

$$H.M = \frac{\theta^6}{\alpha\theta^5+120}\int_0^\infty (\alpha\theta + x^4)e^{-\theta x}dx$$

$$H.M = \frac{\theta^6}{\alpha\theta^5+120}\left(\alpha\theta\int_0^\infty x^{(2)-2}e^{-\theta x}dx + \int_0^\infty x^{(5)-1}e^{-\theta x}dx\right)$$

After simplification of above equation, we obtain

$$H.M = \frac{\theta(\alpha\theta^4 + 24)}{\alpha\theta^5 + 120}$$

4.3 Moment Generating Function and Characteristic Function

Let the random variable X following length biased two parameter Rani distribution with parameters θ and α, then the MGF of X can be obtained as

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_l(x)dx$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f_l(x)dx$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_l(x)dx$$

$$M_x(t) = E(e^{tx}) = \sum_{j=0}^\infty \frac{t^j}{j!} \mu_j'$$

$$M_x(t) = E(e^{tx}) = \sum_{j=0}^\infty \frac{t^j}{j!} \left(\frac{\alpha\theta^5\Gamma(j+2) + \Gamma(j+6)}{\theta^j(\alpha\theta^5 + 120)}\right)$$

$$M_X(t) = \frac{1}{\alpha\theta^5 + 120} \sum_{j=0}^\infty \frac{t^j}{j!\theta^j} \left(\alpha\theta^5\Gamma(j+2) + \Gamma(j+6)\right)$$

Similarly, the characteristic function of length biased two parameter Rani distribution can be obtained as

$$\varphi_x(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\alpha\theta^5 + 120} \sum_{j=0}^\infty \frac{it^j}{j!\theta^j} \left(\alpha\theta^5\Gamma(j+2) + \Gamma(j+6)\right)$$

5. Order Statistics

Order statistics are largely applied in the field of statistical sciences especially in reliability and life testing. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from a continuous population with probability density function $f_x(x)$ and cumulative distribution function $F_X(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) \left(F_X(x)\right)^{r-1} \left(1 - F_X(x)\right)^{n-r} \qquad (10)$$

Using equations (5) and (7) in equation (10), we will obtain the probability density function of r^{th} order statistics of length biased two parameter Rani distribution as

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{\theta^6}{\alpha\theta^5 + 120} x(\alpha\theta + x^4)e^{-\theta x}\right)$$

$$\times \left(\frac{1}{\alpha\theta^5 + 120}\left(\alpha\theta^5\gamma(2, \theta x) + \gamma(6, \theta x)\right)\right)^{r-1}$$

$$\times \left(1 - \frac{1}{\alpha\theta^5 + 120}\left(\alpha\theta^5\gamma(2, \theta x) + \gamma(6, \theta x)\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistic $X_{(n)}$ of length biased two parameter Rani distribution can be obtained as

$$f_{x(n)}(x)=\frac{n\theta^{6}}{\alpha\theta^{5}+120}x(\alpha\theta+x^{4})e^{-\theta x}$$

$$\times\left(\frac{1}{\alpha\theta^{5}+120}\left(\alpha\theta^{5}\gamma(2,\,\theta x)+\gamma(6,\,\theta x)\right)\right)^{n-1}$$

and probability density function of first order statistic $X_{(1)}$ of length biased two parameter Rani distribution can be obtained as

$$f_{x(1)}(x)=\frac{n\theta^{6}}{\alpha\theta^{5}+120}x(\alpha\theta+x^{4})e^{-\theta x}$$

$$\times\left(1-\frac{1}{\alpha\theta^{5}+120}\left(\alpha\theta^{5}\gamma(2,\,\theta x)+\gamma(6,\,\theta x)\right)\right)^{n-1}$$

6. Likelihood Ratio Test

Let X_1, X_2, ……., X_n be the random sample of size n drawn from the two parameter Rani distribution or length biased two parameter Rani distribution. We set up the hypothesis for testing.

$$H_o : f(x)=f(x;\theta,\alpha) \qquad \text{against} \qquad H_1 : f(x)=f_l(x;\theta,\alpha)$$

In order to examine, whether the random sample of size n comes from the two parameter Rani distribution or length biased two parameter Rani distribution, the following test statistic is used.

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\frac{f_l(x;\theta,\alpha)}{f(x;\theta,\alpha)}$$

$$\Delta=\frac{L_1}{L_o}=\prod_{i=1}^{n}\left(\frac{x_i\,\theta(\alpha\theta^{5}+24)}{\alpha\theta^{5}+120}\right)$$

$$\Delta=\frac{L_1}{L_o}=\left(\frac{\theta(\alpha\theta^{5}+24)}{\alpha\theta^{5}+120}\right)^{n}\prod_{i=1}^{n}x_i$$

We should reject the null hypothesis if

$$\Delta=\left(\frac{\theta(\alpha\theta^{5}+24)}{\alpha\theta^{5}+120}\right)^{n}\prod_{i=1}^{n}x_i>k$$

Equivalently, we also reject the null hypothesis where

$$\Delta^{*}=\prod_{i=1}^{n}x_i>k\left(\frac{\alpha\theta^{5}+120}{\theta(\alpha\theta^{5}+24)}\right)^{n}$$

$$\Delta^{*}=\prod_{i=1}^{n}x_i>k^{*},\text{Where }\ k^{*}=k\left(\frac{\alpha\theta^{5}+120}{\theta(\alpha\theta^{5}+24)}\right)^{n}$$

When the sample is large of size n, $2\log\Delta$ is distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus, we refuse to accept the null hypothesis, when the probability value is given by

$p(\Delta^{*}>\lambda^{*})$, Where $\lambda^{*}=\prod_{i=1}^{n}x_i$ is less than a particular level of significance and $\prod_{i=1}^{n}x_i$ is the observed value of the statistic Δ^{*}.

7. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves are known as income distribution curves or oldest classical curves and are applied in different fields like reliability, medicine, insurance and demography. The bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^q xf(x)dx$$

and $L(p)=\frac{1}{\mu_1'}\int_0^q xf(x)dx$

Where $\mu_1'=\frac{2\alpha\theta^5+720}{\theta(\alpha\theta^5+120)}$ and $q=F^{-1}(p)$

$$B(p)=\frac{\theta(\alpha\theta^5+120)}{p(2\alpha\theta^5+720)}\int_0^q\frac{\theta^6}{\alpha\theta^5+120}x^2(\alpha\theta+x^4)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^7}{p(2\alpha\theta^5+720)}\int_0^q x^2(\alpha\theta+x^4)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^7}{p(2\alpha\theta^5+720)}\left(\alpha\theta\int_0^q x^{3-1}e^{-\theta x}dx+\int_0^q x^{7-1}e^{-\theta x}dx\right)$$

After simplification, we obtain

$$B(p)=\frac{\theta^7}{p(2\alpha\theta^5+720)}\left(\alpha\theta\gamma(3,\theta q)+\gamma(7,\theta q)\right)$$

$$L(p)=\frac{\theta^7}{(2\alpha\theta^5+720)}\left(\alpha\theta\gamma(3,\theta q)+\gamma(7,\theta q)\right)$$

8. Entropy

The term entropy is used in various fields such as probability and statistics, physics, communication theory and economics. Entropy discover the diversity, uncertainty, or randomness of a system. Entropy of a random variable *X* is a measure of variation of the uncertainty.

a. Renyi Entropy

The concept of Renyi entropy is important in ecology and statistics as index of diversity. The entropy is named after Alfred Renyi. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int f_l^{\beta}(x)dx\right)$$

Where, $\beta>0$ and $\beta\neq 1$

$$e(\beta)=\frac{1}{1-\beta}\log\int_0^\infty\left(\frac{\theta^6}{\alpha\theta^5+120}x(\alpha\theta+x^4)e^{-\theta x}\right)^\beta dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^\beta\int_0^\infty x^\beta e^{-\theta\beta x}\left(\alpha\theta+x^4\right)^\beta dx\right)$$

Using binomial expansion in above equation, we obtain

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^\beta\sum_{k=0}^\infty\binom{\beta}{k}(\alpha\theta)^{\beta-k}x^{4k}\int_0^\infty x^\beta e^{-\theta\beta x}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^\beta\sum_{k=0}^\infty\binom{\beta}{k}(\alpha\theta)^{\beta-k}\int_0^\infty x^{(\beta+4k+1)-1}e^{-\theta\beta x}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^{\beta}\sum_{k=0}^{\infty}\binom{\beta}{k}(\alpha\theta)^{\beta-k}\frac{\Gamma(\beta+4k+1)}{(\theta\beta)^{\beta+4k+1}}\right)$$

b. Tsallis Entropy

The concept of Tsallis entropy introduced in 1988 by constantino Tsallis as a basis for generalizing the standard statistical mechanics. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty f_l^{\lambda}(x)dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty\left(\frac{\theta^6}{\alpha\theta^5+120}x(\alpha\theta+x^4)e^{-\theta x}\right)^{\lambda}dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^{\lambda}\int_0^\infty x^{\lambda}e^{-\lambda\theta x}(\alpha\theta+x^4)^{\lambda}dx\right)$$

Using binomial expansion in above equation, we get

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^{\lambda}\sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha\theta)^{\lambda-k}x^{4k}\int_0^\infty x^{\lambda}e^{-\lambda\theta x}dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^{\lambda}\sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha\theta)^{\lambda-k}\int_0^\infty x^{(\lambda+4k+1)-1}e^{-\lambda\theta x}dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^{\lambda}\sum_{k=0}^{\infty}\binom{\lambda}{k}(\alpha\theta)^{\lambda-k}\frac{\Gamma(\lambda+4k+1)}{(\lambda\theta)^{\lambda+4k+1}}\right)$$

9. Maximum Likelihood Estimation and Fisher's Information Matrix

In this section, we will estimate the parameters of length biased two parameter Rani distribution by using the method of maximum likelihood estimation and also derive its Fisher's information matrix. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the length biased two parameter Rani distribution, then the likelihood function is given by

$$L(x)=\prod_{i=1}^{n}f_l(x)$$

$$L(x)=\prod_{i=1}^{n}\left(\frac{\theta^6}{\alpha\theta^5+120}x_i(\alpha\theta+x_i^{\,4})e^{-\theta x_i}\right)$$

$$L(x)=\frac{\theta^{6n}}{(\alpha\theta^5+120)^n}\prod_{i=1}^{n}\left(x_i(\alpha\theta+x_i^{\,4})e^{-\theta x_i}\right)$$

The log likelihood function is given by

$$\log L=6n\log\theta-n\log(\alpha\theta^5+120)+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log(\alpha\theta+x_i^{\,4})-\theta\sum_{i=1}^{n}x_i \qquad (11)$$

Now differentiating the log likelihood equation (11) with respect to parameters θ and α. We must satisfy the normal equations as

$$\frac{\partial\log L}{\partial\theta}=\frac{6n}{\theta}-n\left(\frac{5\alpha\theta^4}{\alpha\theta^5+120}\right)+n\left(\frac{\alpha}{(\alpha\theta+x_i^{\,4})}\right)-\sum_{i=1}^{n}x_i=0$$

$$\frac{\partial \log L}{\partial \alpha} = -n\left(\frac{\theta^5}{\alpha\theta^5+120}\right) + n\left(\frac{\theta}{(\alpha\theta + x_i^{\ 4})}\right) = 0$$

Because of the complicated form of above likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore, we use the numerical technique like Newton-Raphson method for estimating the required parameters of the proposed distribution.

In order to obtain the confidence interval, we use the asymptotic normality results. We have that if $\hat{\lambda} = (\hat{\theta},\ \hat{\alpha})$ denotes the MLE of $\lambda = (\theta,\ \alpha)$. We can state the result as follows:

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0,\ I^{-1}(\lambda))$$

Where $I^{-1}(\lambda)$ is Fisher's Information matrix.i.e.

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) \end{pmatrix}$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{6n}{\theta^2} - n\left(\frac{(\alpha\theta^5+120)20\alpha\theta^3 - 5\alpha\theta^4(5\alpha\theta^4)}{(\alpha\theta^5+120)^2}\right) - n\left(\frac{\alpha^2}{(\alpha\theta + x_i^{\ 4})^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{\theta^5(\theta^5)}{(\alpha\theta^5+120)^2}\right) - n\left(\frac{\theta^2}{(\alpha\theta + x_i^{\ 4})^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = -n\left(\frac{(\alpha\theta^5+120)5\theta^4 - \theta^5(5\alpha\theta^4)}{(\alpha\theta^5+120)^2}\right) + n\left(\frac{(\alpha\theta + x_i^{\ 4}) - \alpha\theta}{(\alpha\theta + x_i^{\ 4})^2}\right)$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for θ and α.

10. Application

In this section, we have fitted a real life data set in length biased two parameter Rani distribution to discuss its goodness of fit and the fit has been compared over two parameter Rani, one parameter Rani, exponential and Lindley distributions. The real life data set is given below as:

The following real data set represents 40 patients suffering from blood cancer (leukemia) reported from one of ministry of health hospitals in Saudi Arabia (see Abouammah et al.). The ordered lifetimes (in years) is given below as:

Table 1: Data regarding blood cancer (leukemia) patients reported by Abouammah *et al.* (2000)

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036
2.162	2.211	2.37	2.532	2.693	2.805	2.91	2.912	3.192
3.263	3.348	3.348	3.427	3.499	3.534	3.767	3.751	3.858
3.986	4.049	4.244	4.323	4.381	4.392	4.397	4.647	4.753
4.929	4.973	5.074	5.381					

R software is employed to estimate the unknown parameters along with the model comparison criterion values. In order to compare the length biased two parameter Rani distribution with two parameter Rani, one parameter Rani, exponential and Lindley distributions, we are using the criterion values *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC*

(Akaike Information Criterion Corrected) and *-2logL*.The better distribution is which corresponds to lesser values of *AIC*, *BIC*, *AICC* and *-2logL*. For the calculation of criterion values *AIC*, *BIC*, *AICC* and *-2logL*, following formulas are used:

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \quad \text{and} \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Where k is the number of parameters in the statistical model, n is the sample size and $-2\log L$ is the maximized value of log-likelihood function under the considered model.

Table 2: Performance of fitted distributions

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Length Biased Two Parameter Rani	$\hat{\alpha} = 1.2886827$ $\hat{\theta} = 1.7290884$	$\hat{\alpha} = 1.0556884$ $\hat{\theta} = 0.1334328$	138.7914	142.7914	146.1691	143.1157
Two Parameter Rani	$\hat{\alpha} = 0.3244538$ $\hat{\theta} = 1.4803266$	$\hat{\alpha} = 0.2833934$ $\hat{\theta} = 0.1166114$	141.7743	145.7743	149.152	146.0986
Rani	$\hat{\theta} = 1.36671360$	$\hat{\theta} = 0.07782393$	143.6839	145.6839	147.3728	145.7891
Exponential	$\hat{\theta} = 0.31839887$	$\hat{\theta} = 0.05034278$	171.5563	173.5563	175.2452	173.6615
Lindley	$\hat{\theta} = 0.52692132$	$\hat{\theta} = 0.06074766$	160.5012	162.5012	164.19	162.6064

From results given above in table 2, it can be clearly seen that the length biased two parameter Rani distribution have the lesser *AIC*, *BIC*, *AICC* and *-2logL* values as compared to two parameter Rani, one parameter Rani, exponential and Lindley distributions. Hence, it can be concluded that the length biased two parameter Rani distribution provides better fit as compared over two parameter Rani, one parameter Rani, exponential and Lindley distributions.

11. Conclusion

In the present paper, the new distribution called as length biased two parameter Rani distribution has been studied. The subject distribution is generated by using the length biased technique and taking the two parameter Rani distribution as the base distribution. Its various statistical properties including its mean, variance, survival function, hazard rate function, reverse hazard rate function, moment generating function, characteristic function, order statistics, entropies, bonferroni and lorenz curves have been discussed. Its parameters have also been estimated through the method of maximum likelihood estimation and also its Fisher's information matrix has been observed. Finally, a newly proposed length biased two parameter Rani distribution has been investigated with real life data set to examine its ability and superiority.

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