Class of Accelerated Sequential Procedure For Fixed- Size Confidence Region

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Abstract; **In this paper, a class of accelerated sequential procedures is constructed for fixed size confidence region of parameter in the presence of unknown nuisance parameter under the set -up of distributional relationship developed by Chaturvedi, A., Pandey S.K.,Gupta .,(1991)[2],and results of Hall(1983) [3]was proven .**

Index word: **Accelerated, Chi-Square distribution, Confidence -Region**

1. INTROOUCTION

In order to construct fixed-range confidence interval for a normal mean, assuming the variance to be unknown, Hall [3](1983) proposed an accelerated' sequential Procedure which combines the rates of two-stage and purely sequential procedures and also is more flexible in nature because the number of sampling stages can be reduced only by introducing finite number of observations .Several other experimenters have also developed and studied the same for other distributions also.

In the present Chapter, we develop the classes of 'accelerated' sequential procedures to construct fixed size confidence region for the parameter $\underline{\theta}$ for the bounded risk point estimation. The set up [2] of the problem is:

$\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a random sample of size $n(\geqslant t+1$,from a t variate continuous population, with parameter θ of order $t \times 1$ of interest and Ψ a scalar unknown parameter, let $(\theta', \Psi)' \in R^t \times R^+$. The estimators of θ and Ψ are $\hat{\theta}_n = \hat{\theta}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\hat{\Psi}_n = \hat{\Psi}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. The following hypotheticals are made

(A,): A known positive definite matrix Q, of order t by t, a number $\delta \in (0,1]$ and a positive integer $r \geq 1$ exist ,s . .t. $n[\psi^{-1}(\theta_n - \theta)' Q(\theta_n - \theta)]^\delta \sim \chi^2_{(r)}$

(A_2) : $\hat{\theta}_n$ and $\hat{\Psi}_n$ are independent for all values of n.

(A_3) : For integers $s(\geqslant 1)$,then for all n greater than or equal to s+1,

$r(n-s)\hat{\Psi}_n/\Psi = \sum_{j=1}^{n-s} Z_j^{(r)}$

where $Z_j^{(r)}$'s are iid rv's with $Z_j^{(r)} \sim \chi^2_{(r)}$.[2]

(A_4): $\hat{\Psi}_n$ is a consistent estimator of ψ.

For specified $d(>0)$ and $\delta \in (0,1]$ to construct a confidence region R_n (which may be interval, ellipsoidal or spherical) for θ, of maximum width 2 d and $P(\underline{\theta} \in R_n) \geq \alpha$. We define

$$R_n = \left[\underline{z}: \left\{(\hat{\theta}_n - z) \cdot Q(\hat{\theta}_n - z)\right\}^\delta \leq a^2\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (1.1)$$

when $t \geq 2$, for $\delta = 1, R_n$ is an ellipsoidal confidence region and for $\delta = 1, Q = I_{\text{txt}}, R_n$ reduces to

a spherical region. Moreover, for $t = \delta = 1$ and $l = I_{1\times 1} = 1$, since

$$P(\theta \in R_n) \equiv P\{|\hat{\theta}_n - \underline{\theta}| < d^2\} = p\left\{(\hat{\theta}_n - \theta)^2 \leq d^2\right\},$$

the results based on the region (1.1) are same as those based on a confidence interval of width $2d$.

Denoting by $G^{(r)}(.)$, cdf of a $X_{(r)^2}$ (Chi square)random variable · and utilizing (A_1), we obtain from (1.1)

$$P(\underline{\theta} \in R_n) = P\left[\left\{(\hat{\theta}_n - \underline{\theta})' Q(\hat{\theta}_- - \underline{\theta})\right\}^{\delta} \leq d^2\right]$$
$$= p[X_{(r)}^2 \leq n\psi^{-1}d^2]$$
$$= G^{(r)}(n\psi^{-1}d^2) ------------------------------ (1.2)$$

Let 'a' be the constant satisfying the relation

$G^{(r)}(a^2) = \alpha$.............. (1.3)

Using monotonicity property of distribution function, it follows from (1.2) and (1.3) that, for known ψ; In order to achleve $P(\underline{\theta} \in R_n) \geq \alpha$, the (fixed) sample size required is the smallest positive integer $n \geq n^*$,where

$$n^* = \left(\frac{a}{d}\right)^2 \psi$$

But, in the ignorance of ψ, no fixed sample s1ze procedure achieves the goals of pre-assigned width and coverage probaibiltty' simultaneously for all values of ψ. To meet the requirements, we adopt the following class C_A^{**} of "accelerated "sequential procedures.

Start with a sample of size $m \geq max \cdot \{t+1, s+1\}$, $so\ as\ to$ satisfy $m = o(d^{-2})$ as $d \to 0$ and lim sup. (m/ n*) and fix two constants $L(>0)$ and $\eta \in (0,1)$. start taklng observations' sequentially with the stopping time N_1 defined by

$$N_1 = \text{Inf} \cdot \left[n_1 \geq m : n_1 \geq \eta \left(\frac{a}{d}\right)^2 \hat{\psi}_{n_1}\right]. \qquad (1.4)$$

Based on these N_1 observations compute $\hat{\psi}_{N_1}$. Then jump ahead and collect N_2 observations, where;

$$N_2 = \left[\left(\frac{a}{d} -\right)^2 \hat{\psi}_{N_1} + L\right]^+ + 1 \qquad .... \ (1.5)$$

Let $N = \max \cdot (N_1, N_2)$ and construct the reglon for $\underline{\theta}$.

$$\dot{R}_N = \left[z: \{(\hat{\theta}_N - \underline{z})' \cdot Q(\hat{\theta}_N - \underline{z})\}^{\delta} \leq d^2\right].$$

we first establish some basic lemmas

Lemma 1.1: $\lim_{d\to 0} N_1 = \lim_{d \to 0} N_2 = \infty$

Proof: The proof is an immediate consequence of the definition (1.5) and (1.6) of N_1 and N_2.

Lemma 1.2: $\lim_{d\to 0}(N/n^*) = 1$ a. s ·

proof: From (1.5), we notice that:

$\eta \left(\frac{a}{d}\right)^2 \hat{\psi}_{N_1} \leq N_1 \leq \eta \left(\frac{a}{d}\right)^2 \hat{\psi}_{N_1 - 1} + 1$,.................... (1.7)

or

$(\hat{\psi}_{N_1}/\psi) \leq (N_1{}'\eta n^*) \leq (\hat{\psi}_{N_1} - 1'\psi) + (\eta n^*)^{-1}$.................. .(1.8)

using the fact that $\hat{\psi}_n$ is a consistent estimator of ψ and applying Lemma 1.1, we obtain from (1.8)

$$\lim_{d\to 0}(N_1/\eta^*) = 1 \ \text{ a.s.}$$

similarly using (1.6), it can be shown that

$$\lim_{d\to 0}(N_2/n^*) = 1 \ \text{ a.s.}$$

The lemma now follows from the definition of N.

Lemma 1.3: As $d \to 0, (\eta n^*)^{-\frac{1}{2}}(N_1 - \eta n^*) \xrightarrow{L} N(O, 2q^{-1})$

Proof: Using (A_3), we can rewrite the stopping rule (1.5) as follows

$N_1 = \text{Inf} \cdot \left[n_1 \geq m \sum_{j=1}^{n_1^{-s}} q^{-1} z_j^{(q)} \leq (n_1 - s)\, n_1/\eta n^* \right]$.......................... .(1.9)

Let us define a new stopping variable N_1^* by

$$N_1^* = \text{Inf} \cdot \left[n_1 \geq m - \text{s}: \sum_{j=1}^{n_1} \left(q^{-1} z_j^{(q)}\right) \leq n_1^2 \cdot (1 + \text{s}n_1^{-1})/\eta n^*\right]$$

.... (1.10).

It follows from Lemma 1 of Swanepoel and Vanwyk [4] (1982) that the stopping variables N_1 and N_1^* follows the same probability distribution. Comparing (1.10) with equation (1.1) of Woodroofe [5] (1977), we obtain in his notations $\alpha = 2, \beta = 1, \mu = 1$ and $\tau^2 = 2q$ The lemma now follows from a result of Bhattacharya and Mallik [1] (1973) that

$$(\eta n^*)^{-\frac{1}{2}}(\text{ N}_1 - \eta n^*) \xrightarrow{L} N(O, \beta^2 \tau^2 \mu^{-2})$$

Lemma 1.4: For all m≥ $\{t, s + 2q$-1}, as d tends to 0

$$E(N_1) = \eta n^* + \nu - (s + 2q^{-1}) + o(1),$$

where ν is spectfied.

Proof: In the notations of Woodroofe [5] (1977) a = q/2, $\lambda = \eta n^*, L(n) = 1 + 3n^{-1}$ and L_0 =S The lemma now follows from his Theorem 2.4 that, for all $m > s + 2q^{-1}$ as $d \to 0$

Lemma 1..5: For all $m > \max\{t, s + 2q^{-1}\}$ as $d \to 0$.

$E(N) = n^\star + L - \eta^{-1}(s + 2q^{-1}) + o(1)$, ... (1.11)

$\text{Var} \cdot (N) = (2n^\star/\eta q) + o(d^{-2})$, ... (1.12)

and, for specified $\gamma (> 0)$.

$E|N - E(N)|^\gamma = O(d^{-\gamma})$ ... . (1.13)

Proof: The stopping rule (1.5) can be re-written

$$N_1 = \text{Inf} \cdot \left[n_1 \geq m: q(n_1 - s)\frac{\hat{\psi}_1}{\psi} \leq \frac{d^2 q n_1 (n_1 - s)}{\eta_{a^2})}\right].$$

Let us consider the difference.

$$D_d = \frac{d^2 q N_1 (N_1 - s)}{\eta a^2 \psi} - q_1 (N_1 - s)\frac{\hat{\psi}_1}{\psi} \cdot \qquad (1.14)$$

It follows from Woodroofe [5] (1977) that the mean of the asymptotic distribution of D_d is ν. Let us define.

$$D_d^* = \eta\{q(N_1 - s)\}^{-1}(a/d)^2 \psi D_d$$

Since

$$q(N_1 - s)\frac{\hat{\psi}_{N_1}}{\psi} = \sum_{j=1}^{N_1 - s} z_j^{(q)} \text{ and}$$

$$q(N_1 - s - 1)\frac{\hat{\psi}_{N_{1-1}}}{\psi} = \sum_{j=1}^{N_1 - s - 1} z_j^{(q)},$$

one concludes that;

$$q(N_1 - s)\frac{\hat{\psi}_{N_1}}{\psi} \to q(N_1 - s - 1)\frac{\hat{\psi}_{N_{1-1}}}{\psi}.$$

Hence, from (1.7), (1.14) and (1.15),

$$D_d^* = \eta\{q(N_1 - s)\}^{-1}(a/d)^2\psi\left[\frac{d^2 q(N_1 - a)}{\eta^{\,2}\psi} - q(N_1 - s)\frac{\psi_{N_1}}{\psi}\right]$$

$$\leq \eta\{q(N_1 - s)\}^{-1}(a/d)^2\psi\left[\frac{d^2 q(N_1 - s)}{\eta a^2\psi} - q(N_1 - s - 1)\frac{\hat{\psi}N_1 - 1}{\psi}\right]$$

$$\leq \eta\{q(N_1 - s)\}^{-1}(a/d)^2\psi\left[\frac{d^2 q(N_1 - s)}{\eta a^2\psi} - \frac{q(N_1 - s - 1)}{\psi}\eta^{-1}(d/a)^2\cdot(N_1 - 1)\right]$$

$$= N_1 - (N_1 - S - 1)$$
$$= s + 1$$

Furthermore, assuming (1.7) we get. $= \eta\{q(N_1 - s)\}^{-1}(a/d)^2\psi\left[\frac{d^2 q N_1(N_1 - s)}{\eta a^2\psi} - q(N_1 - s)\frac{\hat{\psi}_{N_1}}{\psi}\right].$

$$\geqslant \eta\{q(N_1 - s)\}^{-1}(a/d)^2\psi\left[\frac{d^2 q^{\,N\,N} N_1 - s)}{\eta a^2\psi} - q(N_1 - s)\frac{\hat{\psi}_{N_1}}{\psi}\right].$$

$$= 0$$

Thus, $0 \leq D_d^* \leq s + 1$ and hence from dominated convergence theorem $E(D_d^*) \to \nu$ as d $\to 0$. Utilizing this result and Lemma 4.9, we obtain for all

$m > \max\cdot\{t, s + 2q^{-1}\}$, as $d \to 0$.

$$E(D_d^*) = \nu = E\left[N_1 - \eta(a/d)^2\hat{\psi}_{N_1}\right]$$

$$E\left[(a/d)^2\hat{\psi}_{N_1}\right] = \eta^{-1}E(N_1 - \nu).$$
$$= n^* - \eta^{-1}(s + 2q^{-1}) + o(1).$$

and (1.11) follows from the definition of N.

By the definition of N,
Var $(N) = \eta^{-2}$Var $\cdot (N_1)$.
1et $n(N_1) = (\eta n^*)^{-1/2}(N_1 - \eta n^*)$.

It follows from theorem2.3 of Woodroofe [5] (1977) that $h^2(N_1)$ is uniformly integrable for all $m > s + 2q^{-1}$.

Now utilizing Lemma 4.8, we obtain for all $m > s + 2q^{-1}$, as $d \to 0$

var. $(N) = \eta^{-2}[2q^{-1}\eta n^*\{1 + o(1)\}]$.
and (1.12) holds
The proof of (1.13) is similar to that of result (3) of Hall (1983).

Result: The main result is now stated and proved in the following theorem:
Theorem: For all m> $(s + 2q^{-1})$,and sufficiently small d, say d≤ d_0

$$P(\underline{\theta} \in R_N) \geq \alpha\,; \qquad if L > \eta^{-1} \quad \{s - (2q)^{-1}\cdot(r - a^2 - 6)\}$$

Proof: Utilizing (A_2), the coverage probability associated with the sampling scheme $(1.5) - (1.6)$ is

$$P(\underline{\theta} \in R_N) = \sum_{n=m}^{\infty} \mathrm{P}\left[n\psi^{-1}\{(\hat{\theta}_n - \underline{\theta})' \cdot Q(\hat{\theta}_n - \underline{\theta})\}\right.$$

$$\left. \leq a^2(n/n^*); N = n\right]$$

$$= \sum_{n=n}^{\infty} G^{(r)}(a^2 n/n^*) \cdot P(N = n)$$

$$= E\left[G^{(r)}(a^2 \text{ N}/n^*)\right].$$

Expanding $G^{(r)}(.)$ around a^2 ' by second-order Taylor's series, we obtain for

$$|a^2 - W| \leq a^2|(N/n^*) - 1|$$

$$P(\theta \in R_N) = G^{(r)}(a^2) + a^2 G^{(r)'}(a^2) E\{(N/n^*)\} + (a^4/2) E\{(N/n^*) - 1\}^2 G^{(r)''(a^2) + \xi d'}$$

where the remainder term $\xi_d = O(a^6 E|N - E(N)|^3)$.

Denoting by $g^{(r)}(.)$, the p.d.f. of a x^2 r.v, we note that

$$G^{(r)'}(x) = g^{(r)}(x)$$

and

$$G^{r''}(x) = \left\{\left(\frac{r}{2} - 1\right) x^{-1} - \frac{1}{2}\right\} g^{(r)}(x).$$

Hence applying Lemma 1.5., we obtain for all

$m >, s + 2q^{-1}$.

$$P\left(\underline{\theta \in R_N}\right) = \alpha + \left(\frac{a^2}{n^*}\right)\{L - \eta^{-1}(s + 2q^{-1}) + o(1)\}$$

$$\cdot g^{(r)}(a^2) + \left(\frac{a^4}{2n^{*2}}\right)\{\text{Var}\,(N) + (E(N) - n^*)^2\}\left\{a^{-2}\left(\frac{r}{2} - 1\right) - \frac{1}{2}\right\} \cdot g^{(r)}(a^2) + O(d^6 d^{-3})$$

$$= \alpha + \left(\frac{a^2}{n^*}\right)\{L - \eta^{-1}(s + 2q^{-1}) g^{(r)}(a^2) + \left(\frac{a^4}{2n^{*2}}\right)\left\{\frac{2n^*}{\eta q} + 0(d^{-2}) + (L - \eta^{-1}(s + 2q^{-1})\right.$$

$$+ \left(\frac{a^4}{2n^{*2}}\right)\left\{\frac{2n^*}{\eta q} + 0(d^{-2}) + (L - \eta^{-1}(s + 2q^{-1}) + \left(\frac{a^4}{2n^{*2}}\right)\right\}\left\{\frac{2n^*}{\eta q} + 0(d^{-2}) + (L - \eta^{-1}(s + 2q^{-1})\right.$$

$$= \alpha + \left(\frac{a^2}{n^*}\right)\left[L - \eta^{-1}(s + 2q^{-1}) + (\eta q)^{-1}\left\{\left(\frac{r}{2} - 1\right) - \frac{a^2}{2}\right\}\right] g(r)(a^2) + 0(d^4) + 0(d^2) + 0(d^3)$$

And the Result follows

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