Convex Conjugate of A Bounded Linear Functional on L^p- Space

[1]T.Srinivasarao, [2]Dr.G.VenkataRao

[1]*Asst. Professor, Department of Mathematics, UCS*
[2]*Asst. Professor University College of Engineering*
[1,2]*Adikavi Nannaya University, Rajahmundr, Andhra Pradesh – 533296 India*

Introduction:

An L^p- space is a normed linear space X and the mapping $\Gamma: X \rightarrow$ R(the set of real numbers) satisfying the linearity property given by $\Gamma(\alpha f + \beta g) = \alpha\,\Gamma(f) + \beta\,\Gamma(g)$ and there is a constant M such that $|\Gamma(f)| \leq M \cdot \|f\|$

Further, the norm of Γ can be defined by $\|\Gamma\| = \sup \frac{|\Gamma(f)|}{\|f\|}$ ...... (1.1)

If in X, $\|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p}$, then $X[0, 1]$ is the space of real valued functions with p to be a non negative real number that satisfies $\frac{1}{p} + \frac{1}{q} = 1$ for some real number q, and all such functions f are in this space $L^p[0,1]$ of $\int_0^1 |f|^p < \infty$. $L^p[0, 1]$ space here after conveniently be called L^p.

Properties of $L^p[0, 1]$:

1.1. $\|\alpha f\|_p \leq |\alpha|\|f\|_p$
1.2. $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ called the Minkowski Inequality
1.3. $\|f\|_p = 0 \leftrightarrow f = \bar{\bar{0}}$
1.4. The precondition $\frac{1}{p} + \frac{1}{q} = 1$ in L^p –space gives that there is another normed linear space $L^q -$ space that realizes $f \epsilon L^p$ and $g \epsilon L^q$ such that $f \cdot g \epsilon L^1[0,1]$ with the property $\int |f \cdot g| \leq \|f\|_p \|g\|_q$

This property is called the Holder's inequality.

Observe that the boundedness of each member $f \epsilon L^p$ by $\int_0^1 |f|^p < \infty$ ascertains that, if $\{f_n\}$ is monotonic and converging to f in L^p, then $\|f_n - f\|_p \rightarrow 0$ which shows that $f \epsilon L^p$ and so, L^p - space is complete leading to $L^p -$ space is a Banach Space. if monotonicity is not considered, then $\{f_n\}$convergence will become pointwise. also, showing the convergence of a Cauchy sequence within the L^p space will confirm the completeness of L^p- space.

The major question of how to realize the suitable $f \in L^p$ for the given $g \in L^q$- space?
For this, there is a two step construction that a sequence of simple functions $\{\varphi_n\}$ and a sequence of continuous functions $\{\psi_n\}$ in view of Littlewood's principles that
$\|f - \varphi_n\|_p < \varepsilon$ and $\|f - \psi_n\|_p < \varepsilon$

Now, the existence of $\Gamma: L^p \rightarrow L^q$ for each $g \in L^q$ that satisfies the relevant $f \in L^p$ such that $\Gamma(f) = \int_0^1 f \cdot g$ with $\|\Gamma\| = \|g\|_q$

Abstract: if $\Gamma: L^p \rightarrow L^q$ is a linear functional having $\frac{1}{p} + \frac{1}{q} = 1$, for each $g \in L^q$ that realizes the relevant $f \in L^p$ such that $\Gamma(f) = \int_0^1 f \cdot g$ with $\|\Gamma\| = \|g\|_q$, then there is a convex complement of Γ given by $\wedge: L^q \rightarrow L^p$ with the property $\alpha\Gamma + \beta\wedge = 1$ for some $0 < \alpha < 1$ and $\beta = 1 - \alpha$. if $\alpha = 0$ or $\alpha = 1$, then the functionals Γand ∧ will become singular and so, will not satisfy the contraction principle. Recollect that, if $\{\varphi_n\}$ converges to f in $L^p -$ space, then $\|\Gamma(f) - \Gamma(\varphi_n)\|_p \leq \|\Gamma\|\|f - \varphi_n\|_p$ and by the properties of the linear functionalΓ, it is known that$\|\Gamma\| = \|g\|_q$. Using this, $\|\Gamma(f) - \Gamma(\varphi_n)\|_p \leq \|g\|_q\|f - \varphi_n\|_p$. since L^p & L^q are linear spaces on the set of real numbers with the symmetric condition $\frac{1}{p} + \frac{1}{q} = 1$, by interchanging the roles of f and g in the Riesz – representation theorem, for each $\wedge: L^q \rightarrow L^p$ and $g \in L^q$, there corresponds a unique $f \in L^p$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $\|\wedge(g) - \wedge(\psi_n)\|_q \leq \|\wedge\|\|g - \psi_n\|_q$

Section 1: ***Norm of the convex conjugate operator***

Theorem 1.1: for each f in L^p, g in L^q, there exists a unique pair of linear functionals $\Gamma: L^p \rightarrow L^q$and $\wedge: L^q \rightarrow L^p$ such that $\Gamma(f) = \int_0^1 fg\, d\mu = \wedge(g)$ with $\|\Gamma\| = \|g\|_q$ and $\|\wedge\| = \|f\|_p$

Proof: there is a sequence of simple functions that estimate $\{\psi_n\}$ that converges to g in L^q space and while L^q- space is complete, $g \in L^q$.

$$\begin{aligned}\wedge(g) &= \int_0^1 gf\,d\mu = \int_0^1 |f|^p\,d\mu \\ &= (\|f\|_p)^p = (\|f\|_p)^{p-1+1} \\ &= (\|f\|_p)^{p-1}(\|f\|_p)^1 \\ &= (\|f\|_p)^{\frac{p}{q}}(\|f\|_p)^1 \\ &= \|g\|_q\|f\|_p\end{aligned}$$

But by (1.1), $\|\wedge\| = \sup\frac{|\wedge(g)|}{\|g\|}$

That means, $\|\wedge\| \geq \frac{|\wedge(g)|}{\|g\|_q}$ or $\wedge(g) \leq \|\wedge\|\|g\|_q$

$\|\wedge\|\|g\|_q \geq \|g\|_q\|f\|_p$

$\|\wedge\| \geq \|f\|_p$

Also, Holder's inequality leads to $|\wedge(g)| = |\int_0^1 gf\,d\mu| \leq \|g\|_q\|f\|_p$

Consequently, $\|\wedge\| \leq \|f\|_p$ and thus $\|\wedge\| = \|f\|_p$

Result: 1.1: by 1.4., $gf \in L^1$ and the indefinite integral $\int_0^t gf\,d\mu$ is a continuous function for $0 \leq t \leq 1$ saying that $\wedge(g)$ is a continuous function on L^1.

Section 2: ***Convexity of the operator*** $\wedge$

Theorem 2.1: if $f \in L^p, g \in L^q$, then $\alpha\Gamma(f) + (1-\alpha)\wedge(g) = \Gamma(f) = \wedge(g)$ for each $0 < \alpha < 1$.

Proof: $\Gamma(f) = \int_0^1 fg\,d\mu = \wedge(g)$, then $\alpha f \in L^p$ for every scalar α, (1- α)$g \in L^q$.

$\int_0^1 \alpha fg\,d\mu = \alpha\,\Gamma(f)$ and $\int_0^1 (1-\alpha)fg\,d\mu = (1-\alpha)\wedge(g)$

So, $\alpha\,\Gamma(f) + (1-\alpha)\wedge(g) = \int_0^1 fg\,d\mu = \Gamma(f) = \wedge(g)$

Result 2.1: $\|\Gamma\| = \|g\|_q$ shows that Γ is a bounded linear functional on L^p while $\int_0^1 [g]^q\,d\mu < \infty$. Also, $\Gamma(f) = \wedge(g)$ for some $f \in L^p \& g \in L^q$. this shows that $\wedge$ is a bounded linear functional on L^q.

Definition 2.1: The bounded linear functional $\wedge$ on L^q is the convex conjugate of Γ on L^p with the property with $\frac{1}{p}+\frac{1}{q}=1$

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