Mutually Singular Subspaces of a Linear Space

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Abstract: A linear space can be made a measurable space'. By taking subsets of the linear space and their linear spans as the elements of $\mathscr{B}$. This work is an exercise to bring out the relationship between the subspaces of a linear space and the respective dimensions as measures that follows the properties of subspaces with respect to the dimension of a subspace or equivalently with respect to the measure. Orthogonal complements in the case of subspaces will be seen as mutually singular measurable subspaces. So, the direct sum of a subspace and its orthogonal complement will be the direct sum of the mutually singular measurable subspaces of a measurable space.

Introduction: if F is a field, $U(F)$ and $V(F)$ are vector spaces, $T: U(F) \rightarrow V(F)$ is a linear transformation, then, the range of T is a subspace of $V(F)$ denoted by $R(T)$ and the Kernel of T is the subspace of $U(F)$ denoted by $N(T)$. The largest subset of independent vectors of a vector space is the basis of the vector space. If the basis has finite number of vectors, then the vector space is finite dimensional and otherwise, infinite dimensional. See that the dimension of a vector space is a non negative integer. The dimension of the range space $R(T)$ is $\rho(T)$ called the rank of T and that of the Kernel or null space $N(T)$ is $\nu(T)$ called the nullity T. If T is a singular transformation, then $\nu(T) > 0$.the dimension theorem says that 'if $T: U(F) \rightarrow V(F)$, then $\rho(T) + \nu(T) = \dim U(F)$. The set of all linear combinations of any subset of either $U(F)$ or $V(F)$ is also a subspace of the respective linear spaces. The set of all linear combinations of a subset S of a linear space $W(F)$ is called the linear span of S denoted by $L(S)$ and is a subspace of $W(F)$. So, the motive of the paper is showing the range space and null space of a linear transformation are mutually singular subspaces with respect to a measure defined in the following sections.

Taking the image vectors of basis of $U(F)$ under T, as the linear combinations of the basis vectors of $V(F)$, whose coefficients are taken into a matrix after getting transposed will indicate the matrix representation of the linear transformation. All the properties of the linear transformation T are satisfied by the matrix of T and at times denoted by matrix A suitable to T. The subspace spanned by the rows of A is the row space and that by the columns is the column space. On the other hand, the row null space is the kernel of T. Since the dimension of the row space and that of column space are equal. So, using the notation 'dim' for dimension of a subspace or a linear space,
Dim column space + dim row null space = dim row space + dim row null space
= number of rows of A = dim $U(F)$.
If the number of rows of A > number of columns of A, then the related linear transformation is singular.

Keywords:
The linear span of a subset S of a linear space $U(F)$ is $L(S)$
$L(S)$ is a subspace of a linear space
$L(S_1) \oplus L(S_2)$ is the direct sum of subspaces
B_S is the standard basis of the linear space
$R(T)$ is the range of T and $P(R(T))$ is the inverse image of $R(T)$ in $U(F)$
$(L(P(R(T))))^t$ is the transpose of the subspace
$\mu(L(S)) = \dim L(S)$: the dimension of the linear space is the measure μ

Section 1: ***Outer measure and measure on a linear space***
There is a subspace $L(P(B_S))$ in $U(F)$ that is isomorphic to $R(T)$ under the inverse transformation P: $R(T) \rightarrow U(F)$ satisfying $L(P(B_S)) = (P(R(T)))^t$ where B_S is the standard basis of $R(T)$ and $L(P(B_S))$ is the linear span of $P(B_S)$ which is the image of $R(T)$ under P. it can be followed that $L(P(B_S))$ and $R(T)$ are isomorphic. This idea in the case of Topology is observed to be the open mapping. Clearly, $L(P(B_S)) \cap N(T) = \{\bar{0}\}$ the trivial subspace, while $L(P(B_S))$ is the pre – image of the non singular part of $R(T)$.
$N(T)$ is the pre – image of singular part $\{\bar{0}\}$ of $V(F)$. The direct sum of $L(P(B_S))$ and
$N(T)$ is $U(F)$. [1]
Further, $L(P(B_S))$ is the orthogonal complement of $N(T)$. i.e.,$L(P(B_S)) \oplus N(T) = U(F)$. [4] & [6].
Definition 1.1: an integer valued function μ: $\mathscr{B} = \{L(S)\} \rightarrow Z^+ \cup \{\bar{0}\}$ is called an integer measure.
Note that, If $U(F)$ is a linear space, $\mathscr{B} = \{L(S)\}$ is the class of subspaces of $U(F)$ for each subset S of $U(F)$, then μ :$\{L(S)\}$ defined by $\mu(L(S)) = \dim L(S)$ [6].
Definition 1.2: if $U(F)$ is a linear space, S is subset, $L(S)$ is a subspace,$\{L(S)\}$is the class of subspaces of $U(F)$, then $\mu(L(S)) = \dim L(S)$ is said to be a measure on $\{L(S)\}$ if
$\mu(\{\bar{0}\}) = 0$ ...... (i)
$\mu(L(S_1 \cup S_2)) \leq \mu(L(S_1)) + \mu(L(S_2))$ ...... (ii)
$L(S_1) \subseteq L(S_2)$ implies $\mu L(S_1) \leq \mu L(S_2)$ ...... (iii)

Observe that the dimension of a subspace or a linear span is an integer greater than or equal to 0. So, μ(L(S)) is a member of $Z^+ \cup \{0\}$. So, μ is well defined. Further, it satisfies Dim $\{\bar{0}\}$ = 0 or μ ($\{\bar{0}\}$) = 0 ...... (i)

Dim $L(S_1 \cup S_2) \leq \dim(L(S_1)) + \dim(L(S_2))$ or,

$\mu(L(S_1 \cup S_2)) \leq \mu(L(S_1)) + \mu(L(S_2))$ ...... (ii)

$S_1 \subseteq S_2$ implies $L(S_1) \subseteq L(S_2)$ implies dim $L(S_1) \leq \dim L(S_2)$ or

$\mu L(S_1) \leq \mu L(S_2)$ ...... (iii)

So, $\mu L(S) = \dim L(S)$ is an outer measure on $\mathscr{B}$. [5]

The above three conditions satisfy the definition of a measure on a measurable space while *U* (*F*) is a measurable space. Note that ρ and ν are also measures of the subspaces namely dimension of the range space and the dimension of the null space. So, conveniently, we write ρ (*T*) = μ(*R* (*T*)) and ν (*T*) = μ (*N* (*T*)).

We now prove that $L(P(B_S))$ and *N* (*T*) are mutually singular with respect to the complementation of subspaces under dimensions treated as measures in the measure space *U* (*F*).[3].

Countable union of linear spans is again a linear span and the intersection of countable number of linear spans is again a linear span confirms that $\mu L(S) = \dim L(S)$ is a saturated measure or a complete measure. [5].

Section 2: ***Signed measure on a linear space***

Definition 2.1: $\mu_1(L(S)) = \mu(L(S) \cap L(P(B_S)))$ and $\mu_2(L(S)) = \mu(L(S) \cap N(T))$

Definition 2.2: If two measures μ_1 & μ_2 are mutually singular, denoted by $\mu_1 \perp \mu_2$ if $\mu_1(L(S)) = 0$ when $L(S) \subseteq L(P(B_S))$ and $\mu_2(L(S)) = 0$ when $L(S) \subseteq N(T)$

Result 2.1: The measures μ_1 & μ_2 are non negative measures on *U* (*F*) when *T*: *U* (*F*) → *U* (*F*) is a linear transformation.

Result 2.2: The measures μ_1 & μ_2 make Radon decomposition on *U* (*F*). [5]

To apply Radon decomposition, the zero measure is taken as the signed measure.

Section 3: ***Mutual Singularity of Subspaces in a linear space***

Definition 3.1: two subspaces $L(S_1)$ and $L(S_1)$ are said to be mutually singular with respect to the measure μ if a non zero vector α of $L(S_1)$ is linearly independent with any set of vectors of $L(S_2)$ and vice versa.

Theorem 3.1: if μ $(L(S_1) + \mu(L(S_2)) = \mu(U(F))$ for some linear space *U* (*F*), and $L(S_1) \cap L(S_2) = \{\bar{0}\}$, then *U* (*F*) = $L(S_1) \oplus L(S_2)$

Result 3.1: A hyperplane through origin and the normal drawn to the plane having foot of the perpendicular at the origin are the mutually singular subspaces in R^n

Result 3.2: Two subspaces that are orthogonal complements in a linear space are mutually singular with respect to the measure equal to the dimension of the linear spaces.

A plane through origin in R^3 is a subspace spanned by 2 linearly independent vectors and so, of dimension 2. The normal drawn to this plane through origin is a straight line spanned by one vector and so, of dimension 1. The plane and its normal meeting at origin are the orthogonal complements whose sum of dimensions is 3 equal to the dimension of the linear space R^3

Result 3.3: Rank ρ and nullity ν are mutually singular measures when *T*: *U*(*F*) → *U*(*F*) [2]

Theorem 3.2: for any subspace *L*(*S*) in a linear space *U*(*F*), there corresponds a unique subspace $L(S_2)^{\perp}$ such that $\mu(L(S) + \mu(L(S_2)^{\perp}) = \mu(U(F))$.

Observation 3.1: Hahn decomposition is not possible with respect to μ(*L*(*S*)) = dim *L*(*S*) while this measure is not a signed measure.

Note: the measure μ in the present discussion is not a complete measure.

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