Error bounds in no normal approximation for the total magnetization in the general Curie-Weiss model

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***Abstract*—the Curie-Weiss model is important in statistical mechanics and has been extensively discussed in the literature. The model not only displays a phase transition, with distinct behaviors at high and low temperature, but also serve as an illustration of various techniques and show how the probabilistic behavior is intimately related to the analytic properties of the thermodynamic potentials (free energy and pressure) of the model. For some history and an overview of first asymptotic results on the Curie-Weiss models, the reader is referred to Ellis and Newman (1978a, 1978b). Using the technique of exchangeable pair approach, Chatterjee and Shao (2011) and Eichelbacher and Löwe (2010) studied a kind of classical Curie-Weiss model. Shao and Zhang (2019) studied a general Curie-Weiss model and got the optimal convergence rate for Kolmogorov bound. In this paper, we establish the Wasserstein bound in nonnormal approximation for the total magnetization in the general Curie-Weiss model at the critical temperature. The proof is based on Stein's method for exchangeable pairs.**

***IndexTerms*—Stein's method,exchangeable pairs, andgeneral Curie-Weiss model.**

1. INTRODUCTION

The Curie–Weiss model has been extensively discussed in the statistical physics field. The asymptotic behavior for the Curie–Weiss model was studied by Ellis and Newman [1–3]. Recently, Stein's method has been used to obtain the convergence rate of the Curie–Weiss model. For example, Chatterjee and Shao [4] and Eichelbacher and Löwe [5] used exchangeable pairs to get a Berry– Esseen bound at the critical temperature of the simplest Curie–Weiss model, where the magnetization was valued on $\{-1,1\}$ with equal probability. More generally, when the magnetization was distributed as a measure ρ with a finite support, Chatterjee and Dey [6] obtained an exponential probability inequality. In this section, we introduce the general Curie–Weiss model.

Let ρ be a probability measure satisfying

$$\int_{-\infty}^{\infty} x\, d\rho(x) = 0 \text{ and } \int_{-\infty}^{\infty} x^2\, d\rho(x) = 1$$

ρ is said to be type k (an integer) with strength λ_ρ if

$$\begin{aligned}&\int_{-\infty}^{\infty} x^j\, d\Phi(x) - \int_{-\infty}^{\infty} x^j\, d\rho(x)\\ &= \begin{cases} 0 & \text{for } j = 0,1,\dots,2k-1, \\ \lambda_\rho > 0 & \text{for } j = 2k \end{cases}\end{aligned} \tag{1}$$

where $\Phi(x)$ is the standard normal distribution function. We define the Curie-Weiss model as follows. For a given measure ρ, let $(X_1, \dots, X_n)$ have the joint probability density function

$$d\mathbb{P}_{n,\beta}(\mathbf{x}) = \frac{1}{Z_n} \exp\left(\frac{\beta(x_1 + \cdots + x_n)^2}{2n}\right) \times \prod_{i=1}^{n} d\,\rho(x_i) \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_n), 0 < \beta \leq 1$ and Z_n is the normalizing constant.

Let ξ be a random variable with probability measure ρ. Assume that $\beta = 1$, there exist constants $b_0 > 0, b_1 > 0$ and $b_2 > 1$ such that

$$\mathbb{E}e^{t\xi} \leq \begin{cases} \exp\left(\frac{t^2}{2} - b_1 t^{2k}\right), & |t| \leq b_0 \\ \exp\left(\frac{t^2}{2b_2}\right), & |t| > b_0 \end{cases} \tag{3}$$

Let $S_n = X_1 + \cdots + X_n$. Ellis and Newman [2], [3] showed that if $\beta = 1$, and ρ is of type k, then $n^{-1+\frac{1}{2k}} S_n$ converges to a nonnormal distribution of a random variable Y with probability density function

$$p(y) = c_1 e^{-c_2 y^{2k}}$$

where $c_2 > 0$ and c_1 is the normalizing constant.

The aim of this paper is to give the Wasserstein distance between $n^{-1+\frac{1}{2k}} S_n$ and Y with optimal rate $\mathrm{C}n^{-\frac{1}{2}}$. In the case of Kolmogorov distance, the reader is refered to [7, Theorem 3.2].

2. Methods

For nonnormal approximation, we will develop Stein's method with the help of exchangeable pairs as follows.

Let $I = (a, b)$ be a interval, where $-\infty \leq a < b \leq \infty$. A function is called regular if f is finite on I and, at any interior point of I, f possesses a right-hand limit and a left-hand limit. Further, f possesses a right-hand limit $f(a+)$ at the point a and a left-hand limit $f(b-)$ at the point b. Let us assume, that the regular density p satisfies the following condition:

(A1) Let p be a regular, strictly positive density on an interval $I = [a, b]$. Suppose p has a derivative p' that is regular on I, has only countably many sign changes, and is continuous at the sign changes. Suppose moreover that

$$\int_I p\,(x)\left|\log\big(p(x)\big)\right|dx < \infty$$

and that

$$\psi(x) := \frac{p'(x)}{p(x)}$$

is regular.

In [8], Stein et. al. proved that a random variable Z is distributed according to the density p if and only if $\mathbb{E}\Big(f'(Z) + \psi(Z)f(Z)\Big) = f(b-)pb—f(a+)p(a+)$ for a suitably chosen class $\mathcal{F}$ of functions f. The corresponding Stein identity is

$$f'(x) + \psi(x)f(x) = h(x) - P(h) \qquad (4)$$

Where h is a measurable function for which

$$\int_I |h(x)|\, p(x)dx < \infty, P(x) := \int_{-\infty}^{x} p\,(y)dy$$

And

$$P(h) := \int_I h\,(y)p(y)dy.$$

The solution $f := f_h$ of this differential equation is given by

$$f(x) = \frac{\int_a^x (h(y) - Ph)\,p(y)dy}{p(x)}$$

For the function $h(x) := 1_{\{x\leq z\}}(x)$ let f_z be the corresponding solution of (4). We will make the following assumptions:

(A2) Let p be a density fulfilling **(A1)**. We assume that for any absolutely continuous function h, the solution f_h of (4) satisfies

$$\|f_h\| \leq c_1\|h'\|, \quad \|f_h'\| \leq c_2\|h'\|$$
$$\text{and } \|f_h''(x)\| \leq c_3\|h'\|,$$

where c_1, c_2 and c_3 are constants.

The Wasserstein bound in Theorem 2 will be a consequence of the following proposition. It is a special case of Theorem 2.4 of Eichelsbacher and Löwe [5].

Proposition 1. *Let Y be a random variable distributed according to p. Let (W, W') be an exchangeable pair, that is, (W, W') and (W', W) have the same joint distribution. Assume that*

$$\mathbb{E}(W - W' \mid W) = \lambda(\psi(W)(W) + R)$$

*where $\psi = p'/p$, $\lambda \in (0,1)$ and R is a random variable. Put $\Delta = W - W'$. Then, under assumption **(A2)**, for any uniformly Lipschitz function h, we obtain*

$$\sup_{\|h'\|\leq 1} |\mathbb{E}h(W) - \mathbb{E}h(Y)| \leq c_2\mathbb{E}\left|1 - \frac{1}{2\lambda}\mathbb{E}(\Delta^2 \mid W)\right| + \frac{c_3}{4\lambda}\mathbb{E}|\Delta|^3 + \frac{c_1}{\lambda}\sqrt{\mathbb{E}(R^2)}.$$

3. Results

The main result is the following theorem. We recall that, throughout this paper, C is a positive constant, and its value may be different for each appearance.

Theorem 2. *Let $(X_1, \ldots, X_n)$ follow the joint probability density function (2), where ρ satisfies (1) and let $W_n = n^{-1+\frac{1}{2k}}S_n$. If $\beta = 1, \rho$ is of type k and (3) is satisfied, then for any uniformly Lipschitz function h, we have*

$$\sup_{\|h'\|\leq 1} |\mathbb{E}h(W) - \mathbb{E}h(Y)| \leq Cn^{-\frac{1}{2k}}$$

where C is a constant depending on b_0, b_1, b_2 and k; the density function of Y is given by

$$p(y) = c_1 e^{-c_2 y^{2k}}, \quad c_2 = \frac{H^{(2k)}(0)}{(2k)!};$$

and c_1 is the normalizing constant and

$$H(s) = s^2/2 - \ln\left(\int_{-\infty}^{\infty} \exp\,(sx)d\rho(x)\right).$$

Proof. We first construct an exchangeable pair (S_n, S_n') as follow. Let $S_n = X_1 + \cdots + X_n$ and let $X' = \{X_1', \ldots, X_n'\}$, where for each i fixed, X_i' is an independent copy of X_i given $\{X_j, j \neq i\}$, i.e., given $\{X_j, j \neq i\}X_i'$ and X_i have the same distribution and X_i' s conditionally independent of X_i (see [5, p.964]). Let I be a random index independent of all others and uniformly distributed over $\{1, \ldots, n\}$. Define $S_n' = S_n - X_I + X_I'$; then (S_n, S_n') is an exchangeable pair (see [7, Theorem 3.2]).

Let $W_n = n^{-1+\frac{1}{2k}} S_n$ and $W_n' = n^{-1+\frac{1}{2k}} S_n'$. Then (W_n, W_n') is also an exchangeable pair. Shao and Zhang [7, Theorem 3.2] proved that

$$\mathbb{E}(W_n - W_n'|W_n) = n^{-2+\frac{1}{k}}\left(\frac{H^{2k}(0)}{(2k-1)!} W_n^{2k-1} + n^{-1+\frac{1}{2k}} R_1\right), \quad (5)$$

and

$$\mathbb{E}\left|1 - \frac{1}{2\lambda}\mathbb{E}[(W_n - W_n')^2|W]\right| \le C n^{-\frac{1}{2k}} \quad (6)$$

where R_1 is a random variable satisying $n^{-1+\frac{1}{2k}}\mathbb{E}|R_1| \le C n^{-\frac{1}{2k}}$ and $\lambda = n^{-2+\frac{1}{k}}$. Next, we consider

$$\begin{aligned}\frac{1}{\lambda}\mathbb{E}|W_n - W_n'|^3 &= \frac{1}{n^{-2+\frac{1}{k}}}\mathbb{E}|X_I - X_I'|^3 \\ &= \frac{1}{n^{-2+\frac{1}{k}}}\mathbb{E}|X_1 - X_1'|^3.\end{aligned}$$

Then,

$$\mathbb{E}|X_1 - X_1'|^3 \le \mathbb{E}|X_1|^3 + 3\mathbb{E}|X_1^2 X_1'| + 3\mathbb{E}|X_1(X_1')^2| + \mathbb{E}|X_1'|^3.$$

Applying Holder's inequality, we have

$$\begin{aligned}\mathbb{E}|X_1^2 X_1'| &\le (\mathbb{E}|X_1|^3)^{2/3}(\mathbb{E}|X_1'|^3)^{1/3} \\ &= \mathbb{E}|X_1|^3\end{aligned}$$

Therefore, by [7, Lemma 5.5 and Lemma 5.6], we have

$$\begin{aligned}\frac{1}{\lambda}\mathbb{E}|W_n - W_n'|^3 &\le \frac{8}{n^{-2+\frac{1}{k}}}\mathbb{E}|X_1|^3 \\ &\le \frac{C}{n^{-2+\frac{1}{k}}}.\end{aligned} \quad (7)$$

Applying **Proposition 1** with (5-7), the proof is completed. □

4. CONCLUSION

In this paper, using the exchangeable pair approach of Stein's method, an optimal convergence rate for nonnormal approximation in the general Curie–Weiss model is achieved. Another version Curie-Weiss is the inhomogeneous Curie-Weiss model with external field, where the inhomogeneity is introduced by adding a positive weight to every vertex and letting the interaction strength between two vertices be proportional to the product of their weights (see [9, 10]). We expect that our results remain true for this model, although if one wants to prove this, extra error terms caused by the approximation with the Curie-Weiss have to be taken into account.

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