BANACH SPACE VALUED SEQUENCE SPACE $l_M(X,p,u)$

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Abstract- **In this paper, we introduce the Banach space valued sequence space $l_M(X,p,u)$ and examine various algebraic and topological properties of it. Finally we introduce a subspace of $l_M(X,p,u)$ and investigate some topological properties of it. Our results generalize and unify the corresponding earlier results of Kamthan and Gupta [3], Ahmad and Bataineh [1].**

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1. Introduction

Lindenstrauss and Tzafriri[6] used the idea of an Orlicz function M to construct the sequence space l_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. The space l_M equipped with the norm

$$\| x \| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a BK space [3, p. 300] usually called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p, 1 \leq p < \infty$. We recall [3, 6] that an Orlicz function M is a function from $[0,\infty)$ to $[0,\infty)$ which is continous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Note that an Orlicz function is always unbounded.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. It is easy to see that always $K > 2$[4]. A simple example of an Orlicz function which satisfies the Δ_2-condition for all values of u is given by $M(u) = a|u|^\alpha (\alpha > 1)$, since $M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u)$. The Orlicz function $M(u) = e^{|u|} - |u| - 1$ does not satisfy the Δ_2-condition.

The Δ_2-condition is equivalent to the inequality $M(lu) \leq K(l)M(u)$ which holds for all values of u, where l can be any number greater than unity.

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x p\,(t)dt$$

where p known as the kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0, p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Before proceeding with the main results we recall [7; second edition] some terminology and notations.

A paranormed space $X = (X,g)$ is a topological linear space in which the topology is given by a paranorm g; a real subadditive function on X such that $g(\theta) = 0, g(x) = g(-x)$ and such that the scalar multiplication is continuous. In the above, θ is the zero in the complex linear space X and continuity of multiplication means that $\lambda_n \to \lambda, x_n \to x$(i.e., $g(x_n - x) \to 0$) imply $\lambda_n x_n \to \lambda x$(i.e., $g(\lambda_n x_n - \lambda x) \to 0$), for scalars λ and vectors x.

A paranorm for which $g(x) = 0$ implies $x = \theta$ is called total paranorm.

A Fr$e$chet space is a complete metric linear space, or equivalently a complete totally paranormed space.

Let w denote the space of all complex sequences $x = (x_k)$. Let X be a linear subspace of w such that X is a Fr$e$chet space with continous coordinate projections. Then we say that X is an FK space, or a Fr$e$chet Koordinat space. If the metric of an FK space X is given by a complete norm then we say that X is a BK space, i.e. a Banach Koordinat space.

A sequence (b_k) of elements of a paranormed space (X,g) is called a Schauder basis for X if and only if, for each $x \in X$, there exists a unique sequence (λ_k) of scalars such that $x = \sum_{k=1}^{\infty} \lambda_k b_k$, i.e., such that $g(x - \sum_{k=1}^{n} \lambda_k b_k) \to 0 (n \to \infty)$.

An FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X. In effect, this means that for each $x = (x_k) \in X$ we have $(x_1, x_2, \dots, x_n, 0,0, \dots) = \sum_{k=1}^{n} x_k e_k \to x (n \to \infty)$, where the convergence is in the metric of X.

Let $(X, \|.\|)$ be a Banach space over the complex field C. Denote by $w(X)$ the space of all X-valued sequences. Let M be an Orlicz function, $u = (u_k)$ be an arbitrary sequence of scalars such that $u_k \neq 0 (k = 1,2, \dots)$ and $p = (p_k)$ be a bounded sequence of positive real numbers.

We now introduce the Banach space valued sequence space $l_M(X,p,u)$ using an Orlicz function M as follows:
$l_M(X,p,u) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} \left[M\left(\frac{\|u_k x_k\|}{\rho} \right) \right]^{p_k} < \infty \; for \; some \; \rho > 0 \right\}$.

Some well-known spaces are obtained by specializing X, M, p and u.

(i) If $X = \mathbb{C}, p_k = u_k = 1$ for all k, then $l_M(X,p,u) = l_M$ (Lindenstrauss and Tzafriri [6]).

(ii) If $X = \mathbb{C}, u_k = 1$ for all k, then $l_M(X,p,u) = l_M(p)$ (Parashar and Choudhary [8]).

(iii) If $X = \mathbb{C}$, then $l_M(X,p,u) = l_M(p,u)$ (Ahmad and Bataineh [1]).

(iv) If $M(x) = x, u_k = 1$ for all k and $p_k = p (1 \leq p < \infty)$ for all k, then $l_M(X,p,u) = l_p(X)$(Leonard [5]).

We denote $l_M(X,p,u)$ as $l_M(X,p)$ when $u_k = 1$ for all k.

In §2, we propose to study various algebraic and topological properties of the sequence space $l_M(X,p,u)$. In §3, certain inclusion relations between $l_M(X,p,u)$ space have been established. In §4, some information on multipliers for $l_M(X,p,u)$ is given. In §5, a subspace of $l_M(X,p,u)$ has been introduced and some topological properties of it has been discussed.

The following inequalities (see, e.g., [7; first edition, p. 190]) are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of positive real numbers. If $H = sup_k p_k$, then for any complex a_k and b_k,

(1.1) $|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$,

where $C = max(1, 2^{H-1})$. Also for any complex λ,

(1.2) $|\lambda|^{p_k} \leq max(1, |\lambda|^H)$.

2. Linear topological structure of $l_M(X,p,u)$ spaces

Theorem 2.1. For any Orlicz function M, $l_M(X,p,u)$ is a linear space over the complex field $\mathbb{C}$. The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. $l_M(X,p,u)$ is a topological linear space, paranormed by

$$g(x) = inf \left\{ \rho^{p_n/G} : \left(\sum_{k=1}^{\infty} \left[M\left(\frac{\| u_k x_k \|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1 \right\} \quad (2.1)$$

where $G = max(1, sup_k p_k)$.

The proof uses ideas similar to those used (e.g.) in [8, p. 421] and the fact that every paranormed space is a topological linear space [9, p. 37].

Theorem 2.3. Let $1 \leq p_k < \infty$, then $l_M(X,p,u)$ is a Frechet space paranormed by (2.1).

Proof. Let (x^i) be a Cauchy sequence in $l_M(X,p,u)$. Let r, u_0 and x_0 be fixed. Then for each $\frac{\epsilon}{r u_0 x_0} > 0$ there exists a positive integer N such that

$$g(x^i - x^j) < \frac{\epsilon}{r u_0 x_0}, for \; all \; \; i,j \geq N.$$

Using definition of paranorm, we get

$\left(\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k^i x_k^i - u_k^j x_k^j\|}{g(x^i - x^j)}\right)\right]^{p_k}\right)^{1/G} \leq 1, for\ all\ i,j \geq N.$

Thus

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k^i x_k^i - u_k^j x_k^j \|}{g(x^i - x^j)}\right)\right]^{p_k} \leq 1\ for\ all\ i,j \geq N.$$

Since $1 \leq p_k < \infty$, it follows that

$$M\left(\frac{\| u_k^i x_k^i - u_k^j x_k^j \|}{g(x^i - x^j)}\right) \leq 1,$$

for each $k \geq 1$ and for all $i,j \geq N$. Hence one can find $r > 0$ with $\frac{u_0 x_0}{2} rp\left(\frac{u_0 x_0}{2}\right) \geq 1$, where p is the kernel associated with M, such that

$$M\left(\frac{\| u_k^i x_k^i - u_k^j x_k^j \|}{g(x^i - x^j)}\right) \leq \left(\frac{u_0 x_0}{2}\right) rp\left(\frac{u_0 x_0}{2}\right).$$

Using the integral representation of Orlicz function M, we get

$$\begin{aligned} \| u_k^i x_k^i - u_k^j x_k^j \| &\leq \frac{r u_0 x_0}{2} g(x^i - x^j) \\ &< \frac{\epsilon}{2}, for\ all\ i,j\ \geq N \end{aligned}$$

Hence $(u^i x^i)$ is a Cauchy sequence in X which implies that (x^i) is Cauchy in X since u is an arbitrary fixed sequence of parameters such that $u_k \neq 0$ for each k. Therefore, for each $\epsilon(0 < \epsilon < 1)$, there exists a positive integer N such that

$\| x^i - x^j \| < \epsilon$ for all $i,j \geq N$.

Now, using continuity of M, we find that

$$\left(\sum_{k=1}^{N}\left[M\left(\frac{\| u_k\left(x_k^i - \lim_{j\to\infty} x_k^j\right) \|}{\rho}\right)\right]^{p_k}\right)^{1/G} \leq 1, for\ all\ i \geq N.$$

Thus

$$\left(\sum_{k=1}^{N}\left[M\left(\frac{\| u_k\left(x_k^i - x_k\right) \|}{\rho}\right)\right]^{p_k}\right)^{1/G} \leq 1, for\ all\ i \geq N.$$

Since N is arbitrary, by taking infimum of such ρ's we get

$$\inf\left\{\rho^{p_n/G} : \left(\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k\left(x_k^i - x_k\right) \|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{G}} \leq 1\right\} for\ all\ i \geq N.$$

Hence $g(x^i - x) < \epsilon$ for all $i \geq N$. That is to say that (x^i) converges to x in the paranorm of $l_M(X,p,u)$. Since $(x^i) \in l_M(X,p,u)$ and M is continous, it follows that $x \in l_M(X,p,u)$.

Corollary 2.4. If p is a constant sequence, then $l_M(X,p,u)$ is a Banach space for $p \geq 1$ and a complete p-normed space for $p < 1$.

Definition 2.5[2] A linear subspace Y of $w(X)$ is a generalized FK space (resp. a generalized BK space) if Y is a Fre$'$chet space (resp. a Banach space) with continous coordinate projections.

In case $X = \mathbb{C}$, then Y becomes an FK space (resp. a BK space).

Theorem 2.6. Let $1 \leq p_k < \infty$, then $l_M(X,p)$ is a generalized FK space paranormed by (2.1).

Proof. In view of Theorem 2.3, it is sufficient to show that the coordinate functionals $P_i: l_M(X,p) \to X$, where $P_i(x) = x_i$ are continuous.

For $\epsilon > 0$ let $\delta > 0$ be such that $0 < \delta < 1$ and $\delta \leq M(\epsilon)$.

Let $g(x) < \delta$ so that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{g(x)}\right)\right]^{p_k} \leq 1$

$$This\ implies\ that\ \sum_{k=1}^{\infty}\left[M\left(\frac{\| x_k \|}{\delta}\right)\right]^{p_k} \leq 1$$

$$and\ so\quad \left[M\left(\frac{\| x_k \|}{\delta}\right)\right]^{p_k} \leq 1\ for\ each\ k \geq 1.$$

As $1 \leq p_k < \infty$, so $M\left(\frac{\|x_k\|}{\delta}\right) \leq 1$ for each $k \geq 1$.
Since $0 < \delta < 1$ and M is convex $\frac{1}{\delta}M(\| x_k \|) \leq M\left(\frac{\|x_k\|}{\delta}\right) \leq 1$ which implies that $M(\| x_k \|) \leq \delta \leq M(\epsilon)$.

Since M is non-decreasing, we have $\| x_k \| < \epsilon$ for each $k \geq 1$ and hence $\| x_k \| < \epsilon$ for each k. Thus the coordinate functionals are continuous and this completes the proof of the theorem.

Corollary 2.7. If p is a constant sequence and $p \geq 1$, then $l_M(X,p)$ is a generalized BK space.

3. Inclusion between $l_M(X,p,u)$ spaces

We now investigate some inclusion relations between $l_M(X,p,u)$ spaces.

Theorem 3.1. If $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers with $0 < p_k \leq q_k < \infty$ for each k, then for any Orlicz function M, $l_M(X,p,u) \subseteq l_M(X,q,u)$.

Proof. Let $x \in l_M(X,p,u)$. Then there exists some $\rho > 0$ such that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k x_k\|}{\rho}\right)\right]^{p_k} < \infty$. This implies that $M\left(\frac{\|u_k x_k\|}{\rho}\right) \leq 1$ for sufficiently large values of k, say $k \geq n_0$ for some fixed $n_0 \in N$. Since M is non-decreasing and $p_k \leq q_k$, we have

$$\sum_{k\geq n_0}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\rho}\right)\right]^{q_k} \leq \sum_{k\geq n_0}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\rho}\right)\right]^{p_k} < \infty.$$

This shows that $x \in l_M(X,q,u)$ and completes the proof.

Theorem 3.2. If $r = (r_k)$ and $t = (t_k)$ are bounded sequences of positive real numbers with $0 < r_k, t_k < \infty$ and if $p_k = min(r_k, t_k)$, $q_k = max(r_k, t_k)$, then for any Orlicz function M, $l_M(X,p,u) = l_M(X,r,u) \cap l_M(X,t,u)$ and $l_M(X,q,u) = G$, where G is the subspace of w generated by $l_M(X,r,u) \cup l_M(X,t,u)$.

Proof. It follows from Theorem 3.1 that $l_M(X,p,u) \subseteq l_M(X,r,u) \cap l_M(X,t,u)$ and that $G \subseteq l_M(X,q,u)$.

For any complex λ, $|\lambda|^{p_k} \leq max(|\lambda|^{r_k}, |\lambda|^{t_k})$, thus $l_M(X,r,u) \cap l_M(X,t,u) \subseteq l_M(X,p,u)$.

Let $A = \{k: r_k \geq t_k\}$ and $B = \{k: r_k < t_k\}$.
If $x = (x_k) \in l_M(X,q,u)$, we write

$$y_k = x_k (k \in A) \quad and \quad y_k = 0 (k \in B); and$$
$$z_k = 0 (k \in A) \quad and \quad z_k = x_k (k \in B).$$

Then since $x = (x_k) \in l_M(X,q,u)$, there exists some $\rho > 0$such that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k x_k\|}{\rho}\right)\right]^{q_k} < \infty$.

Now, $\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k y_k\|}{\rho}\right)\right]^{r_k} = \sum_{k\in A} + \sum_{k\in B} = \sum_{k\in A}\left[M\left(\frac{\|u_k x_k\|}{\rho}\right)\right]^{q_k} < \infty$
and so $y \in l_M(X,r,u) \subseteq G$.

Similarly, $z \in l_M(X,t,u) \subseteq G$.

Thus, $x = y + z \in G$. We have proved that $l_M(X,q,u) \subseteq G$, which gives the required result.

Corollary 3.3. The three conditions $l_M(X,r,u) \subseteq l_M(X,t,u)$, $l_M(X,p,u) = l_M(X,r,u)$ and $l_M(X,t,u) = l_M(X,q,u)$ are equivalent.

Corollary 3.4. $l_M(X,r,u) = l_M(X,t,u)$ if and only if $l_M(X,p,u) = l_M(X,q,u)$.

4. The Space of Multipliers of $l_M(X,p,u)$

For any set $E \subset w(X)$ the space of multipliers of E, denoted by $S(E)$, is given by
$S(E) = \{a = (a_k) \in w(X): ax = (a_k x_k) \in E$ for all $x = (x_k) \in E\}$.
Theorem 4.1. For Orlicz function M which satisfies the Δ_2-condition and Banach algebra X, we have $l_\infty(X) \subseteq S[l_M(X,p,u)]$, where $l_\infty(X) = \{a = (a_k) \in w(X): sup_k \| a_k \| < \infty\}$.

Proof. Let $a = (a_k) \in l_\infty(X)$, $T = sup_k \| a_k \|$ and $x = (x_k) \in l_M(X,p,u)$. Then $\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k x_k\|}{\rho}\right)\right]^{p_k} < \infty$ for some $\rho > 0$. Since M satisfies the Δ_2-condition, there exists a constant $K > 1$ such that

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k a_k x_k \|}{\rho}\right)\right]^{p_k} \le \sum_{k=1}^{\infty}\left[M\left(\frac{\| a_k \|\| u_k x_k \|}{\rho}\right)\right]^{p_k}$$
$$\le \sum_{k=1}^{\infty}\left[M\left((1+[T])\frac{\| u_k x_k \|}{\rho}\right)\right]^{p_k}$$
$$\le \left(K(1+[T])\right)^{H}\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\rho}\right)\right]^{p_k}$$
$$< \infty,$$

where $[T]$ denotes the integer part of T. Hence $a \in S[l_M(X,p,u)]$.

5. A subspace of $l_M(X,p,u)$

In this section we introduce a subspace of $l_M(X,p,u)$ and investigate some topological properties of it.

We define $h_M(X,p,u)$ by

$$h_M(X,p,u) = \left\{x = (x_k) \in w(X): \sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\rho}\right)\right]^{p_k} < \infty for every \rho > 0\right\}.$$

The space $h_M(X,p,u)$ is clearly a subspace of $l_M(X,p,u)$, and the topology is determined by the paranorm of $l_M(X,p,u)$ given by (2.1).

Theorem 5.1. Let $1 \le p_k < \infty$. Then $h_M(X,p,u)$ is a Frechet space with the paranorm given by (2.1).

Proof. Since $h_M(X,p,u)$ is a subspace of $l_M(X,p,u)$ which is a Frechet space under the paranorm given by (2.1), it is sufficient to show that $h_M(X,p,u)$ is closed in $l_M(X,p,u)$. Therefore, let $(x^i) = \left(x_k^i\right)$ be a sequence in $h_M(X,p,u)$ such that $g(x^i - x) \to 0$ as $i \to \infty$, where $x = (x_k) \in l_M(X,p,u)$.

To complete the proof we need to show that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|u_k x_k\|}{\xi}\right)\right]^{p_k} < \infty$ for every $\xi > 0$. To $\xi > 0$ there corresponds an integer m such that $g((x^m - x) < \xi/2)$, and so by the convexity of M,

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\xi}\right)\right]^{p_k} \le \sum_{k=1}^{\infty}\left[\frac{1}{2}M\left(\frac{\| u_k^m x_k^m - u_k x_k \|}{\xi/2}\right) + \frac{1}{2}M\left(\frac{\| u_k x_k \|}{\xi/2}\right)\right]^{p_k}$$
$$\le C\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k^m x_k^m - u_k x_k \|}{g(x^m - x)}\right)\right]^{p_k} + C\sum_{k=1}^{\infty}\left[M\left(\frac{\| u_k x_k \|}{\xi/2}\right)\right]^{p_k}$$
$$< \infty,$$

where $C = max(1, 2^{H-1})$. Thus $x \in h_M(X,p,u)$ which shows that $h_M(X,p,u)$ is complete.

Corollary 5.2. Let $1 \le p_k < \infty$, then $h_M(X,p,u)$ is a generalized FK space with the paranorm given by (2.1).

The proof follows in view of Theorem 2.6 and Theorem 5.1.

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