INVARIANT SUBSPACES OF UNILATERAL SHIFT ON H^2(D)

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Abstract. Let T be the unilateral shift on the Hardy-Hilbert Space on the open unit disk D.
In this paper it is shown that T have no non trivial reducing subspaces and if $M \neq \{0\}$ be an invariant subspace of T then ∃ an inner function φ unique up to the multiplication by a constant of modulus 1 such that $M = \varphi H^2(D)$

Introduction

In this Paper first we define the Hardy-Hilbert space H^2(D), Inner functions and the unilateral shift T on H^2(D).

Then in Theorem 1.1 we showed that H^2(D) is a separable-Hilbert Space.

In Theorem 2.1 we showed T have no non trivial reducing subspaces and in Theorem 2.2 we showed that φH^2(D) is a closed subspace of H^2(D) for all inner functions φ

Then finally we proved our main result that

If $M \neq \{0\}$ be an invariant subspace of T then ∃ an inner function φ unique upto the multiplication by a constant of modulus 1 such that

$$M = \varphi H^2(D)$$

1. The Hardy-Hilbert Space (H^2(D))

1.1. Definition. It is defined as the collection of all the analytic functions on the open unit disk D that have power series representation about the origin.

$$H^2(\mathbb{D} = \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n and \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

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The Inner product of on H^2(D) is defined by

$$< f, g >= \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$

Theorem 1.1. H^2(D) *is Separable Hilbert Space*

Proof. Define a mapping

$$\varphi : l^2(N) \to H^2(D)$$

given by

$$\varphi((an)\infty n=0) = f$$

where $f(z)$ = P∞n=0 $anzn$ Clearly, φ is linear

(1) φ is well defined

Let $(a_n)_{n=0}^{\infty} \in l^2(\mathbb{N}) \Rightarrow \sum_{n=0}^{\infty} |a_n|^2 < \infty$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$$

(2) φ is an isometry

$$||\phi((a_n)_{n=0}^{\infty})||_{H^2} = ||f||_{H^2} = \sqrt{\sum_{n=0}^{\infty} |a_n|^2} = ||(a_n)_{n=0}^{\infty}||_{l^2}$$

(3) φ is surjective

Let $f \in H^2(\mathbb{D}) \Rightarrow f = \sum_{n=0}^{\infty} a_n z^n and \sum_{n=0}^{\infty} |a_n|^2 < \infty$

$\Rightarrow (a_n)_{n=0}^{\infty} \in l^2 \;\; such \;\; that$

$$\phi((a_n)_{n=0}^{\infty}) = f$$

$\therefore$ φ is surjective and an isometry hence unitary

$\therefore l^2(\mathrm{N}) \,'\, H^2(\mathrm{D})$

As l^2 is a separable Hilbert space so is H^2

1.2. $L^2(S^1)$**.** It is defined as the collection of all the square integrable of equivalence classes of square integrable functions with respect to the lesbegue measure normalized such that the measure of whole circle is

1.

Define an inner product on $L^2(S^1)$ by

$$< f, g >= \int_0^{2\pi} f(e^{\iota\theta})\overline{g(e^{\iota\theta})}d\theta$$

$\{e_n : n \in \mathrm{Z}\}$ is an orthonormal basis of $L^2(S^1)$ where

$en(e\iota\theta) = e\iota n\theta$

1.3. $H^{\tilde{}2}(S^1)$**.** It is the closed subspace of $L^2(S^1)$ defined by

$$\{f \in L^2(S^1) :< f,e_n >= 0 \;\forall\; n < 0\}$$

$H^2(\mathrm{D})$ is isomorphic to $H^2(\tilde{}S^1)$ by the mapping

$$H^2(\mathrm{D}) \longrightarrow H^2(\tilde{}S^1)$$

[1]

$$f \rightarrow \bar{f}$$

1.4. **Inner function.** A function $\varphi \in H^2$ is said to be inner if

$$|\varphi\tilde{}| = 1 \; a.e$$

1.5. **invariant Subspace.** A closed subspace M of a Hilbert-Space H is said to be an invariant subspace of an bounded linear operator

T on H if

$$T(M) \subseteq M$$

A invariant subspace M of an bounded linear operator T on a Hilbert space H is said to be **reducing** if

$$T(M) \subseteq M \text{ and } T^?(M) \subseteq M$$

2. Unilateral Shift

2.1. Definition. It is an operator T on H^2(D) defined by $(Tf)(z) = zf(z)$

It is a bounded linear operator on H^2(D) with

$$\|Tf\|_H 2 = \|f\|_H 2$$

And the Hilbert Adjoint $T^?$ is a bounded linear operator on H^2 with

$$(T^\star g)(z) = \frac{g(z) - g(0)}{z} \quad if\, z \neq 0 \quad and \quad (T^\star g)(0) = g'(0)$$

Theorem 2.1. *Only Reducing subspace of the unilateral shift T on H^2(D) is* {0} *and itself*

Proof. Let M be any non zero reducing subspace of T
⇒ M is a closed subspace of H^2(D) such that

$$T(M) \subseteq M \text{ and } T^?(M) \subseteq M \qquad (1)$$

As M is closed ∴ $H^2(D = M\,^{M} M^{\perp}$

If $1 \in M$

$$\Rightarrow (T1)(z) = z \in M \Rightarrow (T^2 1)(z) = z^2 \in M$$

$$\Rightarrow (T^n 1)(z) = z^n \in M$$

In this way M contains all the orthonormal basis of H^2(D)

$$\Rightarrow M = H^2(D)$$

Similarly if $1 \in M^{\perp} \Rightarrow M^{\perp} = H^2$(D) which is not true as M is non zero
Now Suppose if possible

$$1 = f + g$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in M$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in M^{\perp}$ are two non zero non constant functions

$$\Rightarrow T^?(f + g) = T^? 1 = 0$$

$$\Rightarrow (a_1 + b_1) + (a_2 + b_2) + \dots\dots\dots = 0$$

$$\Rightarrow a_n = -b_n \,\forall\, n \geq 1$$

Then by (1) we have

$$T^? f \perp T^? g$$

$$\Rightarrow |a_1|^2 + |a_2|^2 + \dots\dots\dots\dots = 0$$

$$\Rightarrow a_n = b_n = 0 \,\forall n \geq 1$$

Which shows that f and g are constant functions which is a contradiction hence

$$M = H^2(D)$$

∴ The only reducing subspaces of T are {0} and H^2(D)

Theorem 2.2. *Let $\varphi \in H^2$(D) be an inner function then φH^2(D) is a closed subspaces of H^2(D)*

Proof. Define a mapping

$$B : H^2(D) \to H^2(D)\; f \to \varphi f$$

Clearly B is linear

(1) B is an isometry

$$||Bf||^2_{H^2} = ||\phi f||^2_{H^2} = ||\tilde{\phi f}||_{\tilde{H^2}} = \int_0^{2\pi} |\tilde{\phi}|^2 |f(\tilde{e^{\iota}\theta})|^2 d\theta = ||\tilde{f}||^2_{\tilde{H^2}} = ||f||_{H^2}$$

∴ B is an isometry

(2) $RangeB = \varphi H^2(D)$ is closed

Let $g \in \overline{RangeB}$. Then ∃ a sequence (g_n) in $RangeB$ such that

$$g_n \to g \text{ as } n \to \infty$$

As $g_n \in RangeB$ ∀ n Then for each n ∃ $f_n \in H^2(D)$ such that

$$Bf_n = g_n \ \forall \ n$$

AS (g_n) is a convergent sequence, hence cauchy

Then For any $\epsilon > 0$ ∃ a positive integer N such that

$$||g_n - g_m||_{H^2} < \epsilon \ \forall \ n \geq N$$

As B is an isometry

$$\therefore ||f_n - f_m||_{H^2} = ||Bf_n - Bf_m||_{H^2} = ||g_n - g_m||_{H^2} < \epsilon \ \forall \ n \geq N$$

⇒ (f_n) is a cauchy sequence hence convergent since $H^2(D)$ is a Hilbert Space.

∴ ∃ $f \in H^2(D)$ such that

$$f_n \to f \text{ as } n \to \infty$$

As B is a bounded operator hence continuous

$$\Rightarrow Bf_n \to Bf \text{ as } n \to \infty$$

$$\Rightarrow g_n \to Bf \text{ as } n \to \infty$$

By the uniqueness of limits we have

$$Bf = g$$

$$\Rightarrow g \in RangeB \therefore RangeB = \varphi H^2(D) \text{ is closed}$$

3. Classification of Invariant subspaces of unilateral shift

Theorem 3.1. *Let $M \neq 0$ be any invariant subspace of unilateral shift on $H^2(D)$ Then ∃ an inner function $\varphi \in H^2(D)$ such that $M = \varphi H(D)$*

Proof. Let M be a non zero invariant subspace of T

$$\text{if } M = H^2(D) \Rightarrow \varphi \equiv 1$$

if $M \neq H^2(\mathbb{D})$

(1) $T(M) \subsetneq M$

Suppose if possible $T(M) = M$

As M 6= {0} Then ∃ 0 6= $f \in M$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, Then as $T(M) = M$

$$\Rightarrow \exists \ g_1 \in M \text{ such that}$$

$$(Tg_1)(z) = zf(z)$$

$$\Rightarrow a_0 = 0$$

Similarly, $\exists\ g_2 \in M$ such that

$$(Tg_2)(z) = zg_1(z)$$

$$\Rightarrow a_1 = 0$$

Continuing in this way we get

$$a_0 = a_1 = a_2 = \ldots\ldots\ldots = 0$$

$$\Rightarrow f = 0$$

which is a contradiction

$$\therefore T(M) \$ M$$

As T is an isometry

$\therefore T(M)$ is closed

$\Rightarrow M = T(M)^{M} N$ *where* N 6= {0}

Then we can choose $\varphi \in N$ such that $||\varphi||_H 2 = 1$

(2) φ is inner

As

$$\phi \in N \subseteq T(M)^{\perp} \ \ and \ \ T(M) \subseteq M$$

$$\Rightarrow < \phi, T^k\phi >= 0 \Rightarrow < \tilde{\phi}, T^{\tilde{k}}\phi >= 0 \ \ \forall \ \ k \geq 1$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \phi(\tilde{e^{\iota\theta}})\overline{\phi(\tilde{e^{\iota\theta}})}e^{-\iota k\theta} = 0 \ \ \forall \ \ k \geq 1 \qquad (1)$$

Taking conjugate we have

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(\tilde{e^{\iota\theta}})\overline{\phi(\tilde{e^{\iota\theta}})}e^{\iota k\theta} = 0 \ \ \forall \ \ k \geq 1 \qquad (2)$$

As

$$||\phi||_{H^2} = 1 \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \phi(\tilde{e^{\iota\theta}})\overline{\phi(\tilde{e^{\iota\theta}})} = 1$$

$$\therefore < |\tilde{\phi}|^2, e_n >= 0 \ \ \forall \ \ n = \pm 1. \pm 2, \ldots\ldots. \ \ and \ \ < |\tilde{\phi}|^2, e_0 >= 1$$

$$|\phi(\tilde{e^{\iota\theta}})| = 1 \ \ a.e.$$

$\therefore \varphi$ is an inner function

(3) $N = span(\varphi)$

Suppose $\exists\ \psi \in N$ such that $||\psi||_H 2 = 1$ and $< \psi,\varphi >= 0$ Then

By above claim ψ is also an inner function

$$\therefore \phi f \ \ and \ \ \psi f \in H^2(\mathbb{D}) \ \ \forall \ \ f \in H^2(\mathbb{D})$$

$$Let \ \ \phi(z) = \sum_{n=0}^{\infty} a_n z^n \ \ and \ \ \psi(z) = \sum_{n=0}^{\infty} b_n z^n$$

For each $n,m \in$ N Define

$$\phi_n(z) = \sum_{k=0}^{n} a_k z^k \ \ and \ \ \psi_m(z) = \sum_{k=0}^{m} b_k z^k$$

$$\Rightarrow < \phi_m, \psi_n > = \sum_{j,k=0} a_j \overline{b_k} < z^j, z^k > = 0 \ \forall \ m, n \in \mathbb{N}$$

$$As \ \ \phi_m \to \phi \ \ and \ \ \psi_n \to \psi \ \ as \ \ n, m \to \infty$$

$$\Rightarrow < \phi_m \psi, \psi_n \phi > \to 0 \ \ as \ \ n, m \to \infty$$

$$\Rightarrow ||\phi\psi||_{H^2} = 0$$

Which is a contradiction since

$$|\varphi(e^{\imath\theta})| = 1 \ a.e$$

$$and$$

$$|\psi(e^{\imath\theta})| = 1 \ a.e$$

(4) $M = \varphi H^2(\mathrm{D})$

Since

$$\varphi \in M$$

and M is an invariant subspace of T

$$\Rightarrow (T\varphi)(z) = z\varphi(z) \in M$$

$$\Rightarrow (T^2\varphi)(z) = z^2\varphi(z) \in M$$

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Continuing in this way we have

$$\{\varphi(z), z\varphi(z), z^2\varphi(z), ...\} \subseteq M$$

As M is closed

$$\therefore \varphi H^2(\mathrm{D}) \subseteq M$$

By Theorem 2.2 $\varphi H^2(\mathrm{D})$ is a closed subspace of M

$$\Rightarrow M = \varphi H^2(\mathrm{D})^{\mathrm{M}} \varphi H^2(\mathrm{D})^{\perp} = T(M)^{\mathrm{M}} span(\varphi)$$

Suppose if possible $\exists \ \psi \in M$ such that

$$\psi \perp \varphi H^2(\mathrm{D}) \Rightarrow \psi \perp \varphi$$

$$\Rightarrow \psi \in T(M)$$

Let

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n$$

As $\psi \in T(M) \Rightarrow \exists \ \psi_1 \in T(M)$ such that

$$(T\psi_1)(z) = z\psi_1(z)$$

$$\Rightarrow b_0 = 0$$

Similarly $\exists \ \psi_2 \in T(M)$ such that

$$(T\psi_2)(z) = z\psi_1(z)$$

$$\Rightarrow b_1 = 0$$

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continuing in this way we get

$$\psi \equiv 0$$

$$\therefore M = \varphi H^2(\mathrm{D})$$

Theorem 3.2. *(Uniqueness) Let $M \neq \{0\}$ be an invariant subspace of T then ∃ an inner function φ unique upto the multiplication by a constant of modulus 1 such that*

$$M = \varphi H^2(\mathrm{D})$$

Proof. Let M be any non zero invariant subspace of unilateral shift T on $H^2(\mathrm{D})$
Then By Theorem 3.1 ∃ an inner function φ such that $M = \varphi H^2(\mathrm{D})$
Let

$$M = \varphi H^2(\mathrm{D}) = \psi H^2(\mathrm{D})$$

Then By Theorem 3.1

$$N = span\varphi = span\psi$$

⇒ ∃ an constant c such that

$$\varphi = c\psi$$

As

$$|\varphi(e^{\imath\theta})| = 1 = |\psi(e^{\imath\theta})|\ a.e.$$

$$\Rightarrow |c| = 1$$

Hence Proved

Conclusion

In this Paper we classify all the invariant subspaces of unilateral shift on the $H^2(\mathrm{D})$
i.e. if M is any reducing subspace of T the eithet $M = \{0\}$ or M is whole of $H^2(\mathrm{D})$. Also if M be an invariant but not reducing subspace of T then ∃ unique inner function φ such that

$$M = \varphi H^2(\mathrm{D})$$

References

[1] Praveen Sharma, "HARDY SPACES ON THE DISK AND ITS APPLICATIONS", International Journal of Creative Research Thoughts (IJCRT), ISSN:2320-2882, Volume.8, Issue 3, pp.1980-1986, March 2020
[2] Kreyszig, Erwin. Introductory functional analysis with applications. Vol. 1. New York: wiley, 1978.