# A review on-Characterization of Involute & Evolute curve in 4-dimensional space

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***Abstract*: In this study a kind of generalized involute and evolute curve-couple is considered in 4-dimensional Euclidean space. The necessary and sufficient condition for the curve possessing generalized involute as well as evolute is obtained. Recently, extensive research has been done on evolute curves in Minkowski space-time. However, the special characteristics of curves demand advanced level observations that are lacking in existing well-known literature. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space.**

***Index Terms*: Involute curve, Evolute Curve, Frenet curve, Mate curve, Carten null curve, Minkowski space**

## I. INTRODUCTION

Many mathematicians did work about the general theory of the curves in Euclidean space. Now, we have much understanding on their local geometry as well as their global geometry. Identification of a regular curve is one of the important and interesting complication in the theory of curves in Euclidean space. In differential geometry an evolute is the envelope of the normals of the specific curve.

An evolute and its involute are defined in mutual pairs. The evolute of any curve is defined as the locus of the centers of curvatures of the curve. The original curve is then described as the involute of the evolute. Evolutes and involutes (also known as evolvents) were studied by C. Huygens .In the theory of curves, one of the important and interesting problems is the characterization of regular curves, in particular, the involute–evolute of a given curve. Evolutes and involutes (also known as evolvents) were studied by C. Huygens . According to D. Fuchs, an involute of a given curve is a curve to which all tangents of the given curve are normal. He also defined the equation for an enveloping curve of the family of normal planes for a space curve. Suleyman and Seyda determined the concept of parallel curves, which means that if the evolute exists, then the evolute of the parallel arc will also exist and the involute will coincide with the evolute. Brewster and David stated that a curve is composed of two arcs with a common evolute, and the common evolute of two arcs must be a curve with only one tangent in each direction. In general, the evolute of a regular curve has singularities, and these points correspond to vertices. Emin and Suha determined that an evolute Frenet apparatus can be formed by an involute apparatus in four-dimensional Euclidean space, so, in this way, another orthonormal of the same space can be obtained. Shyuichi Izumiya defined evolutes as the loci of singularities of space-like parallels and geometric properties of non-singular space-like hyper surfaces corresponding to the singularities of space-like parallels or evolutes. Takami Sato investigated the singularities and geometric properties of pseudo-spherical evolutes of curves on a space-like surface in three-dimensional Minkowski-space. Marcos Craizer stated that the iteration of involutes generates a pair of sequences of curves with respect to the Minkowski metric and its dual. According to Boaventura Nolasco and Rui Pacheco, correspondence between plane curves and null curves in Minkowski three-space exists. He also described the geometry of null curves in terms of the curvature of the corresponding plane curves. M. Turgut and S. Yilmaz obtained the Frenet apparatus of a given curve by defining the space-like involute–evolute curve couple in Minkowski space-time. Some researchers have investigated evolute curves and their characterization in Minkowski space as well as in Euclidean space. Many researchers have dealt with evolute–involute curves, but no research has been carried out on the Cartan null curve. In this study, a special kind of generalized evolute and involute curve is considered in four-dimensional Minkowski space. We obtained necessary and sufficient conditions for the curve possessing a generalized evolute as well as an involute.

In this paper we consider Evolute curves in Euclidean space with respect to casual character of the plane spanned by tangent and the first binormal of the curve. In this paper, a kind of generalized evolute and involute curve is considered for Cartan null curve in Minkowski space-time. The necessary and sufficient conditions for a curve possessing generalized evolute as well as involute mate curves is obtained.

## II. PRELIMINARIES

Consider the Minkowski space-time ($E^4_1$, H) where $E^4_1 = \{z = (z_1 , z_2 , z_3 , z_4)/ z_i$ € R } and $H = -dz^2_1 + dz^2_2 + dz^2_3 + dz^2_4$ . For any U = (x1 , x2 , x3 , x4) and V = (y1 , y2 , y3 , y4) € $T_zE$, .We denote U . V = H (U , V ) = -x1y1 + x2y2 + x3y3 + x4y4.

Let I be an open interval in R and G : I → $E^4_1$ be a regular curve in $E^4_1$ parameterized by arc length parameter s and {T ,N1 ,N2 ,N3 } be a moving Frenet frame along G, consisting of tangent vector T, principal normal vector N1, the first binormal vector N2 and the second binormal vector N3 respectively, so that T ^ N1 ^ N2 ^ N3 coincides with the standard orientation of $E^4_1$ . From Frenet seret formula

$$\begin{pmatrix} T' \\ N1' \\ N2' \\ N3' \end{pmatrix} = \begin{pmatrix} 0 & \text{€}2K1 & 0 & 0 \\ -\text{€}1K1 & 0 & \text{€}3K2 & 0 \\ 0 & -\text{€}2K2 & 0 & -\text{€}1\text{€}2\text{€}3K3 \\ 0 & 0 & -\text{€}3K3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N1 \\ N2 \\ N3 \end{pmatrix}$$

Where H(T,T)= €1, H(N1,N1) =€2 , H(N2,N2)= €3 , H(N3 ,N3) =€4 , €1€2€3€4= -1 , $\text{€}_i \in \{-1, 1\}$ , i $\in \{1,2,3,4\}$
In Specific, the succeeding conditions exist:
H (T, N1) =H (T, N2) =H (T, N3) = H (N1, N2) = H (N1, N3) = H (N2, N3) =0

A curve G(s) in $E^4{}_1$ can be spacelike, timelike , or null if its velocity G'(s) are commonly spacelike , timelike , or null. A null curve G is parametrized by pseudo-arc s if H (G"(s), G" (s)) =1. Furthermore nonnull curve G, we have this condition H(G'(s) , G'(s)) = ±1 . If G is a null Carten curve then the Carten Frenet frame is given by,

$$\begin{pmatrix} T' \\ N1' \\ N2' \\ N3' \end{pmatrix} = \begin{pmatrix} 0 & K1 & 0 & 0 \\ K2 & 0 & -K1 & 0 \\ 0 & -K2 & 0 & K3 \\ K3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N1 \\ N2 \\ N3 \end{pmatrix}$$

Where K1(s) = 0 if G(s) is a null straight line or K1(s)=1 in all other cases.
In this case
T.T= N2.N2=0 N1.N1=N3.N3=1 T.N2 =1 T.N1 = T. N3=N1.N2=N1.N3=N2.N3=0

At any point of G , the plane spanned by { T, N2} is called the ( 0,2) tangent plane of G.The plane spanned by { N1, N3} is called the ( 1 , 3) normal plane of G.

Let G: I $\rightarrow$ $E^4$ & G* : I $\rightarrow$ $E^4{}_1$ be two regular curves in $E^4{}_1$ where s is the arc -length parameter of G. Denote s* = f( s) to be the arc – length parameter of G*. For Any s ∈ I , if the (0,2) tangent plane of G at G(s) of coincides with the (1,3) normal plane at G*(s) of G* then G* is called (0 , 2 ) involute curve of G in $E^4{}_1$ & G is called the ( 1,3) evolute curve of G* in $E^4{}_1$

### 2.1 The (0,2)-Involute Curve of a Given Curve in $E^4{}_1$

In this section, we proceed to study the existence and expression of the (0,2)-involute curve of a given curve in $E^4{}_1$.

**Theorem 1**. Let G : I → $E^4{}_1$ be a regular curve parameterized by arc-length s so that k1, k2 and k3 are not zero.
If G* possesses the (0,2)-involute mate curve, G*(s) = G(s) + ($\Phi_0$ -s)T(s) + $\psi$B1(s), with $\psi \neq 0$, then k1, k2 and k3 satisfy

$$\frac{k2}{k1} = \tau \quad , \quad \frac{k3}{k1} = t1\,(\tau + \text{€}1\,\text{€}2\ t2)\ ,\ \tau = \frac{\Phi_0 - s + \psi\, t1^2 t2}{\psi\,(1 - \text{€}1\text{€}3 t1^2)}$$

Where $\Phi_0, \psi$,$t_1$ & $t_2$ are given constants. Moreover, the three curvatures of G* are given by

$$K_1{*} = -\frac{\text{€}1\text{€}4\text{€}2^* f t3^2}{\psi(\tau + \text{€}1\text{€}3 t2)} \qquad K_2{*} = \frac{f(\text{€}4\text{€}3^* t2\,\tau - \text{€}2\text{€}3^* t2^2 - \text{€}1\text{€}4^* t3^2)}{\psi\, t1(\tau + \text{€}1\text{€}3 t2)} \qquad K_3{*} = -\frac{\text{€}4\text{€}4^* f}{\psi t1}$$

Where f ≠0 The associated frenet frame are given by
T* = f t3 ( t1 N +B2 ) , N* = f (T+t2B1) , B1* =gt3(-N+t1B2) , B2*= f (-t2T + B1)

**Theorem 2.** Let G :I → $E^4{}_1$ be a regular curve with arc-length parameter s so that k1, k2 and k3 are not zero.
If a possesses the (0,2)-involute mate curve G*(s) = G(s) + ($\Phi_0$ -s)T(s) + $\psi$B1(s), then k1 and k2 satisfy gk1 + f k2=0 where $\Phi_0$,f & g are constants.Moreover , the 3 curvatures of G* are given by

$$K_1{*} = \frac{1}{\text{€}1\text{€}2\text{€}2^* f(s - \Phi_0)} \qquad K_2{*} = \frac{-\text{€}4\text{€}3^* k3}{\text{€}2(s - \Phi_0)k1} \qquad K_3{*} = \frac{\text{€}4\text{€}1^* f k3}{\text{€}2(s - \Phi_0)k1}$$

### 2.2 The (1,3)-Envolute Curve of a Given Curve in $E^4{}_1$

**Theorem 3**. Let G : I → $E^4{}_1$ be a regular curve parameterized by arc-length s so that k1, k2 and k3 are not zero.
If G* possesses the (1,3)-envolute mate curve, G*(s) = G(s) + $\frac{1}{i\,k1(s)}[iN(s) + j\,B2(s)] - \frac{1}{k3(s)}$ B2(s) then k1, k2 and k3 satisfy $\text{€}1ik1 + \text{€}3(jk2 - jk3)$=0 where i & j are given constants. Three curvatures of G* are given by

$$k1{*} = -\text{€}1\text{€}2{*}\,\frac{\sqrt{2}\ (ik1)}{f\prime}\ , \qquad k2{*} = \frac{\sqrt{2}\ \left(\frac{\text{€}4\text{€}3^* k3}{2i} - \text{€}1\text{€}4^* jk1\right)}{f\prime} \qquad k3{*} = -\sqrt{2}\,k3/(2if')\ ,\ f' = (1/ik1)$$

The associated Frenet frames are given by,
T*= i N + j B2, N*= (T + B1)/$\sqrt{2}$ , B1*=-j N+i B2, B2*=(-T+B1)/$\sqrt{2}$

**Theorem 4**. Let G : I → $E^4_1$ be a regular curve parameterized by arc-length s so that k1, k2 and k3 are not zero.
If G* possesses the (1,3)-envolute mate curves, G*(s) = G(s) + $\frac{1}{i\,k1(s)}[iN(s) + j\,B2(s)]$ then k2 and k3 satisfy ik2 – jk3=0 where i & j are given constants. Moreover, the Three curvatures of G* are given by
k1*= -€1€2* i k1 / f' , k2*= -€1€2* j k1 / f' , k3*= -€3$i^{-1}$ k3/ f' , f'= (1/ik1)
The associated Frenet frames are given by, T*= i N + j B2 , N*=T , B1*=-j N+i B2, B2*=B1

### 2.3 The (1,3)-Evolute Curve of a Cartan Null Curve in $E^4_1$

In this section, we proceed to study the existence and expression of the (1,3)-evolute curves of a given Cartan null curve in $E^4_1$. At any point of G, the plane spanned by {N, B2} is called the (1,3)-normal plane of G.

Let G : I → $E^4_1$ & G* : I → $E^4_1$be two regular curves in $E^4_1$, where s is the arc-length parameter of G. Denote s* = f (s) to be the arc-length parameters of G*. For any s ∈ I, if the (0,2)-tangent plane of G at G(s) coincides with the (1,3)-normal plane at G*(s) of G*, then G* is called the (0,2)-involute curve of G in $E^4_1$ and G is called the (1,3)-evolute curve of G* in $E^4_1$.

**Theorem 5**. Let G : I → $E^4_1$ be a null Cartan curve with arc length parameter s so that k1 = 1, and k2 & k3 are not zero.
If G possesses the (1,3)-envolute mate curve, G*(s) = G(s) + $\frac{1}{i(s)}[iN(s) + j\,B2(s)] - \frac{1}{k3(s)}$ B2(s) then k1, k2 and k3 satisfy i+ik2-jk3=0 where i & j are given constants. Three curvatures of G* are given by
k1*= $\frac{\sqrt{2}\ i}{f'}$, k2*= $\frac{\sqrt{2}\ ((\frac{k3}{2i})-j)}{f'}$ k3*= - $\sqrt{2}$ k3/(2if') , f'= (1/i)
The associated Frenet frames are given by,
T*= i N + j B2, N*= (T + B1)/√2 , B1*=-j N+i B2, B2*=(-T+B1)/√2

**Theorem 6**. Let G : I → $E^4_1$ be a null Cartan curve with arc length parameter s so that k1 = 1, and k2& k3 are not zero.
If G possesses the (1,3)-envolute mate curve, G*(s) = G(s) + $\frac{1}{ik1(s)}[iN(s) + j\,B2(s)]$ then k2 and k3 satisfy ik2-jk3=0 where i & j are given constants. Three curvatures of G* are given by
k1*= -i / f' , k2*= -j / f' k3*= $i^{-1}$ k3/ f' , f'= (1/i)
The associated Frenet frames are given by,
T*= i N + j B2, N*= T , B1*=-j N+i B2, B2*=B1

**Theorem 7**. Let G : I → $E^4_1$ be a null Cartan curve with arc length parameter s so that k1 = 1, and k2 & k3 are not zero.
If G possesses the (1,3)-envolute mate curve, G*(s) = G(s) + $\frac{1}{i(s)k1}[iN(s) + j\,B2(s)] - \frac{1}{k3(s)}$ B2(s) then k1, k2 and k3 satisfy ik1+i-jk3=0 where i & j are given constants. Three curvatures of G* are given by
k1*= $\frac{\sqrt{2}\ ik1}{f'}$, k2*= $\frac{\sqrt{2}\ ((\frac{k3}{2i})-jk1)}{f'}$ k3*= - $\sqrt{2}$ k3/(2if') , f'= (1/ik1)
The associated Frenet frames are given by,
T*= i N + j B2, N*= (T + B1)/√2 , B1*=-j N+i B2, B2*=(-T+B1)/√2

**Case 2:** For t = 0, we have the following theorem.

**Theorem 8**. Let G : I → $E^4_1$ be a null Cartan curve with arc length parameter s so that k1 = 1, and k2 & k3 are not zero.
If G possesses the (1,3)-envolute mate curve, G*(s) = G(s) + $\frac{1}{i(s)k1}[iN(s) + j\,B2(s)]$) then k1, k2 and k3 satisfy ik2-jk3=0 where i & j are given constants. Three curvatures of G* are given by
k1*= -i / f' , k2*= -j / f' k3*= $i^{-1}$ k3/ f' , f'= (1/i)
The associated Frenet frames are given by,
T*= i N + j B2, N*=T, B1*=-j N+i B2, B2*=B1

## III.Conclusions

This paper established new kinds of generalized evolute and involute curves in four-dimensional Minkowski space by providing the necessary and sufficient conditions for the curves possessing generalized evolute and involute curves. Furthermore, the study invoked a new type of (1,3)-evolute and (0,2)-evolute curve in four-dimensional Minkowski space. The study also provided a new kind of generalized null Cartan curve in four-dimensional Minkowski space. For this new type of curve,the study provided several theorems with necessary and sufficient conditions and obtained significant results. The understanding of evolute curves with this new type evolute curve in four-dimensional Minkowski space will be beneficial for researchers in future studies. The designing of a framework for the involutes of order k of a null Cartan curve in Minkowski spaces will be considered in future work.

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