Unreliable Retrial $M^X/G/1$ Queueing Systems under Exhaustive and Non-exhaustive Vacations with Feedback Services

[1]A. Pon Kiruthiga, [2]M.I. Afthab Begum

[1]M.Phil Scholar, [2]Professor
[1]Department of Mathematics,
[1]Avinashilingam Deemed University, Coimbatore, Tamil Nadu, India

***Abstract*: A retrial batch arrival single server queueing system is considered with generally distributed service time. Whenever the system becomes empty, the server takes vacation following the Multiple Adapted Vacation (MAV) policy. The MAV policy generates all the other server vacations including the non-vacation case. The present paper also assumes that the server may take Bernoulli schedule single vacation in between two consecutive services for pre-preparatory work or maintenance of the system. The system is subjected to unpredictable breakdowns and the service interrupted customers resume their service from the point of interruption as soon as the system is fixed. Also, a customer who is not satisfied with the service may demand for instantaneous re-services. The system is analysed under steady-state and the probability generating functions of the queue size, the queue size probabilities and mean queue lengths when the system is in different states are obtained. The performance measures of the system are also inspected numerically. Decomposition property for vacation is established and the results of various vacation models are derived as particular cases.**

***Index Terms*: Retrial, Multiple Adapted vacation, Bernoulli Schedule vacation, breakdown, feedback.**

I. INTRODUCTION

The mathematical results on retrial queues were first published in 1950's. Yang and Templeton (1987) surveyed the works on the retrial queues. Later fundamental methods and results on the retrial queues are found in the detailed study of Falin and Templeton (1997) and Artalejo (1999) gave a bibliography on this topic. Artalejo and Atencia (2004) have worked on the batch arrival retrial queues with single server. Takacs (1963) introduced the feedback of the customers when the service is not satisfactory to the customers. The present model deals with the infinite feedback where the customer may claim for the repetition of the service for infinite number of times until the customer gets satisfied with his service. Kalidass and Kasturi (2013) considered a model describing the transaction in ATM machine where the customers after performing a transaction can feedback immediately for the next transaction not moving to the tail of the queue. Another application of feedback queues is Automative Repeat re-Quest (ARQ) protocol, a high frequency communication network.

Server vacation model was first discussed by Levy and Yechiale (1975). Various authors analysed the vacation queueing models by considering different types of vacation as independent characteristics. Baba (1986) considered the batch arrival queue with multiple vacations. Aissani (1998) discussed the retrial $M^X/G/1$ queueing models with exhaustive vacations. Then, Choudhury (2002) modelled batch arrival queueing system with a single vacation. Multiple vacation retrial models were analysed by Krishna Kumar and Pavai Madheswari (2003). Later a new vacation policy for $M^X/G/1$ queueing system where the server may leave for atmost J-vacations was proposed by Ke and Chu (2006) and Ke et al. (2010). The Multiple Adapted Vacation policy was considered by Mytalas and Zazanis (2015). Nawel ARRAR et al (2017) explained the decomposition property for retrial single server model. Keilson and Servi (1986) introduced the Bernoulli scheduled vacation i.e. vacation between services. Gaver (1962) seems to be the first to study the effect of server breakdowns for $M^X/G/1$ queues. The breakdowns are assumed to occur only when the server is busy and are independent of each other. Keilson (1962), Yue and Tu (2001) and many others have contributed a lot to the queueing models with server breakdown. Fiems et al. (2008) fixed the probability for repeat / resumption of service whereas Krishnamoorthy et al. (2009) provided specific rule to decide whether to repeat / resume an interrupted service. The most recent works on queueing models with interruptions may be found in the survey paper of Krishnamoorthy et al. (2012).

II. MATHEMATICAL ANALYSIS OF THE SYSTEM

Model Description

Arrival Pattern: The customers arrive in batches following Poisson process with batch arrival rate λ. The batch size $\mathbf{X}$ of the arriving customers has the probability distribution $\Pr(X = k) = g_k,\ k = 1,2,\ldots$ where $\sum_{k=1}^{\infty} g_k = 1$. There is no waiting space in front of the server. If the arriving customers find the server free, then one of the arriving customers begins the service and the others leave the service area and enters a queue of blocked customers called orbit in accordance with FCFS queue discipline; whereas if the server is busy or on vacation then all the customers join the orbit. The retrial times are generally distributed random variables with distribution function $\mathbf{A(t)}$.

Multiple Adapted Vacation (MAV) Policy: Cycle starts when the system becomes empty and the server is deactivated. During this idle period, the server may leave the system for a vacation with probability $\boldsymbol{\delta_0}$ or remains idle in the system with probability $\mathbf{1} -$

$\boldsymbol{\delta_0}$. Upon returning from the first vacation, the server joins the system to begin a busy period, if he finds at least one customer in the retrial orbit. Otherwise if the server finds the system still empty, he either takes a second vacation with the probability $\boldsymbol{\delta_1}$ or joins the system and remains idle with the probability $\mathbf{1-\delta_1}$. This process continues until a batch of customers arrives and then starts a busy cycle. Thus, the vacation policy is determined by the sequence of the probabilities $\{\boldsymbol{\delta_j}\}$ where $j = 0,1,\ldots$ The vacation time of the server is independent and identically distributed with the distribution function $\mathbf{VI(t)}$, the corresponding density function $\mathbf{vI(t)}$ and LST $\mathbf{VI^*(\theta)}$.

Busy Period: Busy period starts immediately when at least one customer enters the empty system or the customer in the head of the orbit retries when the server is idle. During busy period the server serves the customers one by one. The busy period ends when the system becomes empty again.

Feedback policy: At the end of each service, if the customers are not satisfied by the service they may claim for re- service with probability **f**, or leave the system with probability **1-f**. Thus instantaneous re-service is provided to the customers on their demand and is termed as feedback services. Such feedback services continue until the customer is satisfied and leave the system. The service time of the regular or feedback services are arbitrarily distributed with distribution functions $\mathbf{S_i(t)}$, the corresponding density function $\mathbf{s_i(t)}$ and LST $\mathbf{S_i^*(\theta)}$ for $i = 0$ or 1.

Breakdown Period: The server may breakdown at any time during regular or feedback services with Poisson rates $\boldsymbol{\alpha_0}$ or $\boldsymbol{\alpha_1}$ respectively. Then the server is immediately sent for repair and the service is stopped for a while. The interrupted service will be resumed as soon as the server returns to the channel after the repairs being rectified. The repair times of the server are arbitrarily distributed with distributions $\mathbf{R_i(t)}$ and density functions $\mathbf{r_i(t)}$ for $i = 0$ or 1 according as the breakdowns occur during regular or feedback services.

Bernoulli Schedule Vacation: After completing a service to a customer and sending him out of the system, the server may take at most one vacation before starting a new regular service to the next customer, with probability **p** or prefer to continue to serve the next customer with probability **1-p**. The busy vacation time of the server is assumed to be independent identically distributed random variables with common distribution $\mathbf{VB(t)}$, density function $\mathbf{vB(t)}$ and LST $\mathbf{VB^*(\theta)}$.
The model is denoted by **$\mathbf{M^X}$/G/1/MAV/breakdown/feedback(∞)/BSV**. If f(x) is the density function of the probability distribution F(x), then the LST is given by $F^*(\theta) = \int_0^\infty e^{-\theta x}\, d(F(x))$.

Let $A^0(t), VI^0(t), VB^0(t), S_0^0(t), S_1^0(t), R_0^0(t)$ and $R_1^0(t)$ respectively denote the remaining times of the random variables namely retrial time, vacation time during idle and busy period, primary and feedback service time and repair time during primary and feedback service at time t. Further different states of the server at time t are designated by $Y(t) = \{0,1,2,3,4,5,6\}$ which respectively denotes idle state, vacation during idle and busy period, primary and feedback busy state, repair during primary and feedback service. The supplementary variables are introduced in order to obtain a bivariate Markov process $\{N(t), \delta(t)\}$ where N(t) denotes the queue size random variable and $\delta(t) = \left(A^0(t), VI^0(t), VB^0(t), S_0^0(t), S_1^0(t), R_0^0(t), R_1^0(t)\right)$ according as $Y(t) = (0,1,2,3,4,5,6)$ respectively.
Let $PI_n(w,t)dt = Pr\{N(t) = n, w \le A^0(t) \le w + dt, Y(t) = 0\}$ be the joint probability that at time t, there are n customers in the retrial orbit and the server is idle with the remaining retrial time between w and w + dt, where $n \ge 1$ and let $PI_0(t) = Pr\{N(t) = 0, Y(t) = 0\}$ represents the probability that the server is idle at time t and there is no customer in the retrial orbit.
Let $QI_{n,k}(x,t)dt = Pr\{N(t) = n, x \le VI^0(t) \le x + dt, Y(t) = 1\}$ be the joint probability that at time t, there are n customers in the retrial orbit and the server is in k^{th} vacation with the remaining vacation time between x and x + dt, where $n \ge 0$.
Let $QB_n(x,t)dt = Pr\{N(t) = n, x \le VB^0(t) \le x + dt, Y(t) = 2\}$ denote the joint probability that at time t, there are n customers in the retrial orbit and the server is under busy vacation with the remaining vacation time between x and x + dt, where $n \ge 1$.
Let $P_n^0(x,t)dt = Pr\{N(t) = n, x \le S_0^0(t) \le x + dt, Y(t) = 3\}$ represents the joint probability that at time t, there are n customers in the retrial orbit and the server is busy under primary service to the customer with the remaining service time between x and x + dt, where $n \ge 0$.
Let $P_n^1(x,t)dt = Pr\{N(t) = n, x \le S_1^0(t) \le x + dt, Y(t) = 4\}$ be the joint probability that at time t, there are n customers in the retrial orbit and the server is busy under feedback service to the customer with the remaining service time between x and x + dt, where $n \ge 0$.
Let $BR_n^0(x,y,t)dt = Pr\{N(t) = n, S_0^0(t) = x, y \le R_0^0(t) \le y + dt, Y(t) = 5\}$ denote the joint probability that at time t, there are n customers in the retrial orbit, the remaining primary service time is x and the server is under repair during primary service to the customer with the remaining repair time between y and y + dt, where $n \ge 0$.
Let $BR_n^1(x,y,t)dt = Pr\{N(t) = n, S_1^0(t) = x, y \le R_1^0(t) \le y + dt, Y(t) = 6\}$ be the joint probability that at time t, there are n customers in the retrial orbit, the remaining feedback service time is x and the server is under repair during feedback service to the customer with the remaining repair time between y and y + dt, where $n \ge 0$.
At steady state, as $t \to \infty$, the queue size probabilities are assumed to be independent of time. Let $PI_n(w)$, $QI_{n,k}(x)$, $QB_n(x)$, $P_n^0(x)$, $P_n^1(x)$, $BR_n^0(x,y)$ and $BR_n^1(x,y)$ respectively denote the steady-state probabilities and $PI_n(\theta)$, $QI_{n,k}(\theta)$, $QB_n(\theta)$, $P_n^0(\theta)$, $P_n^1(\theta)$, $BR_n^0(\theta,\theta')$ and $BR_n^1(\theta,\theta')$ be the corresponding LST of the probabilities. Various stochastic processes involved in the queueing system are assumed to be independent of each other. The equations under the steady state condition are analyzed using Supplementary Variable Technique.

LST of the Steady State Queue Size Equations

$$\lambda PI_0 = \sum_{j=1}^{\infty}(1-\delta_j)\, QI_{0,j}(0) + (1-\delta_0)(1-f)\left(P_0^0(0)+P_0^1(0)\right) \quad (1)$$

$$\theta PI_n^*(\theta) - PI_n(0) = \lambda PI_n^*(\theta) - \sum_{j=1}^{\infty} QI_{n,j}(0)A^*(\theta) - (1-p)(1-f)\left(P_n^0(0)+P_n^1(0)\right)A^*(\theta) - QB_n(0)A^*(\theta), \quad n \ge 1 \quad (2)$$

$$\theta QI_{0,1}^*(\theta) - QI_{0,1}(0) = \lambda QI_{0,1}^*(\theta) - \delta_0(1-f)\left(P_0^0(0)+P_0^1(0)\right)VI^*(\theta) \quad (3)$$

$$\theta QI_{0,k}^*(\theta) - QI_{0,k}(0) = \lambda QI_{0,k}^*(\theta) - \delta_{k-1}QI_{0,k-1}(0)VI^*(\theta), \qquad k \ge 2 \quad (4)$$

$$\theta QI_{n,k}^*(\theta) - QI_{n,k}(0) = \lambda QI_{n,k}^*(\theta) - \lambda\sum_{i=1}^{n} QI_{n-i,k}^*(\theta)g_i\,, \qquad n \ge 1, k \ge 1 \quad (5)$$

$$\theta P_n^{0*}(\theta) - P_n^0(0) = (\lambda+\alpha_0)P_n^{0*}(\theta) - \lambda S_0^*(\theta)\sum_{k=1}^{n}\int_0^{\infty} PI_{n-k+1}(w)dw\, g_k(1-\delta_{0,n}) - PI_{n+1}(0)S_0^*(\theta)$$
$$-\lambda\sum_{k=1}^{n} P_{n-k}^{0*}(\theta)g_k(1-\delta_{0,n}) - \lambda PI_0 g_{n+1}S_0^*(\theta) - BR_n^{0*}(\theta,0), \qquad n \ge 0 \quad (6)$$

$$\theta P_n^{1*}(\theta) - P_n^1(0) = (\lambda+\alpha_1)P_n^{1*}(\theta) - f\left(P_n^0(0)+P_n^1(0)\right)S_1^*(\theta) - \lambda\sum_{k=1}^{n} P_{n-k}^{1*}(\theta)g_k(1-\delta_{0,n}) - BR_n^{1*}(\theta,0), \quad n \ge 0 \quad (7)$$

$$\theta QB_n^*(\theta) - QB_n(0) = \lambda QB_n^*(\theta) - p(1-f)\left(P_n^0(0)+P_n^1(0)\right)VB^*(\theta) - \lambda\sum_{k=1}^{n-1} QB_{n-k}^*(\theta)g_k(1-\delta_{1,n}), \qquad n \ge 1 \quad (8)$$

$$\theta' BR_n^{i**'}(\theta,\theta') - BR_n^{i*}(\theta,0) = \lambda BR_n^{i**'}(\theta,\theta') - \lambda\sum_{k=1}^{n} BR_{n-k}^{i**'}(\theta,\theta')g_k(1-\delta_{0,n}) - \alpha_i P_n^{i*}(\theta)R_i^{*'}(\theta'), \quad n \ge 0, i = 0,1 \quad (9)$$

The following partial generating functions are introduced to study the model.

$PI^*(z,\theta) = \sum_{n=1}^{\infty} PI_n^*(\theta)z^n \qquad PI(z,0) = \sum_{n=1}^{\infty} PI_n(0)\, z^n$

$QI_k^*(z,\theta) = \sum_{n=0}^{\infty} QI_{n,k}^*(\theta)z^n \qquad QI_k(z,0) = \sum_{n=0}^{\infty} QI_{n,k}(0)z^n \qquad k \ge 1$

$P^{i*}(z,\theta) = \sum_{n=0}^{\infty} P_n^{i*}(\theta)z^n \qquad P^i(z,0) = \sum_{n=0}^{\infty} P_n^i(0)z^n \qquad i = 0,1$

$QB^*(z,\theta) = \sum_{n=1}^{\infty} QB_n^*(\theta)z^n \qquad QB(z,0) = \sum_{n=1}^{\infty} QB_n(0)z^n$

$BR^{i**'}(z,\theta,\theta') = \sum_{n=0}^{\infty} BR_n^{i**'}(\theta,\theta')z^n \qquad BR^{i*}(z,0,0) = \sum_{n=0}^{\infty} BR_n^{i*}(0,0)z^n \qquad i = 0,1$

Probability Generating Functions

The partial probability generating functions of the queue size probabilities at an arbitrary epoch when the system is in different states are obtained through algebraic manipulation and are listed below:

$$BR^{0**'}(z,0,0) = \alpha_0(1-f)\left(P_0^0(0)+P_0^1(0)\right)Q_R(z)\frac{1-S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{h_{\alpha_0}(w_x(z))}\frac{1-R_0^{*'}(w_x(z))}{w_x(z)} \quad (10.1)$$

$$BR^{1**'}(z,0,0) = \alpha_1 f(1-f)\left(P_0^0(0)+P_0^1(0)\right)Q_R(z)\frac{S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)}\frac{1-S_1^*\left(h_{\alpha_1}(w_x(z))\right)}{h_{\alpha_1}(w_x(z))}\frac{1-R_1^{*'}(w_x(z))}{w_x(z)} \quad (10.2)$$

$$P^{0*}(z,0) = (1-f)\left(P_0^0(0)+P_0^1(0)\right)Q_R(z)\frac{1-S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{h_{\alpha_0}(w_x(z))} \quad (10.3)$$

$$P^{1*}(z,0) = f(1-f)\left(P_0^0(0)+P_0^1(0)\right)Q_R(z)\frac{S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)}\frac{1-S_1^*\left(h_{\alpha_1}(w_x(z))\right)}{h_{\alpha_1}(w_x(z))} \quad (10.4)$$

$$QB^*(z,0) = p(1-f)\left(P_0^0(0)+P_0^1(0)\right)\left(Q_R(z)\frac{(1-f)S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)} - 1\right)\frac{1-VB^*(w_x(z))}{w_x(z)} \quad (10.5)$$

$$QI^*(z,0) = (1-f)\left(P_0^0(0)+P_0^1(0)\right)\sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\frac{1-VI^*(w_x(z))}{w_x(z)} \quad (10.6)$$

$$PI^*(z,0) = (1-f)\left(P_0^0(0)+P_0^1(0)\right)Y_R(z)\frac{1-A^*(\lambda)}{\lambda} \quad (10.7)$$

where $w_x(z) = \lambda(1-X(z))$; $h_{\alpha_i}(w_x(z)) = w_x(z) + \alpha_i\left(1-R_i^{*'}(w_x(z))\right) \qquad i = 0,1$

$$Q_R(z) = -\frac{w_x(z)}{z-HBV_R^*(w_x(z))}IVR(z) \quad (11)$$

$$IVR(z) = A^*(\lambda)\frac{\varphi}{\lambda} + M_1(z)IV_0(z) \quad (12)$$

$$IV_0(z) = \sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\frac{1-VI^*(w_x(z))}{w_x(z)} + p\frac{VB^*(w_x(z))-1}{w_x(z)} \quad (13)$$

with $VI^*(w_x(z)) = \sum_{n=0}^{\infty}\beta_n z^n$; $\beta_n = \int_0^{\infty} e^{-\lambda t}\sum_{i=0}^{\infty}\frac{(\lambda t)^i g_n^{(i)}}{i!}d(vI(t))$

[where $g_n^{(i)}$ = Pr{there are n customers in the orbit at the end of k^{th} batch arrival}]

$$HBV_R^*(w_x(z)) = \frac{(1-f)S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)}\left(1-p+pVB^*(w_x(z))\right)M_1(z) \quad (14)$$

$$M_1(z) = A^*(\lambda) + X(z)(1-A^*(\lambda)) \quad (15)$$

$$Y_R(z) = (1-f)Q_R(z)\frac{S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)}\left(1-p+pVB^*(w_x(z))\right) - I_0(z) \quad (16)$$

$$I_0(z) = \varphi + \sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\,(1-VI^*(w_x(z))) + p(VB^*(w_x(z))-1) \quad (17)$$

The PGF of queue size probabilities during busy period and idle period are respectively given by

$$P_{\mathbf{B}}(z) = \sum_{i=0}^{1} BR^{k**'}(z,0,0) + \sum_{i=0}^{1} P^{k*}(z,0) = (1-f)\left(P_0^0(0)+P_0^1(0)\right)\frac{Q_R(z)}{w_x(z)}\left[1 - \frac{(1-f)S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)}\right]$$

and $P_{idle}(z) = QB^*(z,0) + QI^*(z,0) + PI^*(z,0) + PI_0 = (1-f)\left(P_0^0(0)+P_0^1(0)\right)\frac{Q_R(z)}{w_x(z)}\left[\frac{(1-f)S_0^*\left(h_{\alpha_0}(w_x(z))\right)}{1-fS_1^*\left(h_{\alpha_1}(w_x(z))\right)} - z\right]$

Thus the total probability generating function $(PIF_R(z))$ of the queue size probabilities is

$$PIF_R(z) = \sum_{i=0}^{1} BR^{k**'}(z,0,0) + \sum_{i=0}^{1} P^{k*}(z,0) + QB^*(z,0) + QI^*(z,0) + PI^*(z,0) + PI_0$$

$$PIF_R(z) = (1-f)\left(P_0^0(0) + P_0^1(0)\right)Q_R(z)\frac{1-z}{w_x(z)} \tag{18}$$

III. PERFORMANCE MEASURES

In this section, the performance measures such as the steady state queue size probabilities and the mean queue size of the proposed model at different states are calculated. The following results derived from equations (11) to (17) are used to obtain the performance measures.

$$Q_R(1) = \frac{\lambda E(X)}{1-\rho} IVR(1)\,;$$

$$Q_R'(1) = \frac{\lambda}{1-\rho}\left[IVR'(1)E(X) + \frac{IVR(1)}{2}\left(E\big(X(X-1)\big) + \frac{E(X)}{1-\rho}\left(\lambda E\big(X(X-1)\big)E(HBV_R) + \big(\lambda E(X)\big)^2 E(HBV_R^2)\right)\right)\right]$$

$$IVR(1) = A^*(\lambda)\frac{\varphi}{\lambda} + \sum_{k=0}^{\infty}\beta_0^k \prod_{i=0}^{k}\delta_i\, E(VI) - pE(VB)\,;$$

$$IVR'(1) = \lambda E(X)\left[\sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\left(\frac{1-A^*(\lambda)}{\lambda}E(VI) + \frac{E(VI^2)}{2}\right) - p\left(\frac{1-A^*(\lambda)}{\lambda}E(VB) + \frac{E(VB^2)}{2}\right)\right]$$

$$IV_0(1) = \sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\, E(VI) - pE(VB)\;; \quad IV_0'(1) = \frac{\lambda}{2}E(X)\left(\sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\, E(VI^2) - pE(VB^2)\right)$$

$$[HBV_R^*(w_x(z))]_{z=1} = 1\,;\quad [HBV_R^*(w_x(z))]'_{z=1} = \lambda E(X)\left(E(H_0) + \frac{f}{1-f}E(H_1) + pE(VB) + \frac{1-A^*(\lambda)}{\lambda}\right) = \rho\;;$$

$$[HBV_R^*(w_x(z))]''_{z=1} = \lambda E\big(X(X-1)\big)E(HBV_R) + \big(\lambda E(X)\big)^2 E(HBV_R^2)$$

where $E(HBV_R) = E(H_0) + \frac{f}{1-f}E(H_1) + pE(VB) + \frac{1-A^*(\lambda)}{\lambda}$

and $E(HBV_R^2) = \alpha_0 E(S_0)E(R_0^2) + E(S_0^2)(1+\alpha_0 E(R_0))^2 + \alpha_1\frac{f}{1-f}E(S_1)E(R_1^2) + \frac{f}{1-f}E(S_1^2)(1+\alpha_1 E(R_1))^2 + 2\frac{f}{1-f}E(H_0)E(H_1)$
$+2\left(\frac{f}{1-f}\right)^2\big(E(H_1)\big)^2 + pE(VB^2) + 2pE(VB)\left(E(H_0) + \frac{f}{1-f}E(H_1)\right) + 2\left(E(H_0) + \frac{f}{1-f}E(H_1) + pE(VB)\right)\frac{1-A^*(\lambda)}{\lambda}$

with $E(H_i) = E(S_i)\big(1+\alpha_i E(R_i)\big) \qquad i = 0,1$

$$M_1(1) = 1\,;\quad M_1'(1) = E(X)\big(1-A^*(\lambda)\big)\,;\quad M_1''(1) = E\big(X(X-1)\big)\big(1-A^*(\lambda)\big)$$

$$Y_R(1) = Q_R(1) - \varphi\,;\quad Y_R'(1) = Q_R'(1) + \lambda E(X)Q_R(1)\left(E(H_0) + \frac{f}{1-f}E(H_1) + pE(VB)\right) - I_0'(1)$$

$$I_0(1) = \varphi\,;\quad I_0'(1) = \lambda E(X)\left(pE(VB) - \sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\, E(VI)\right)$$

Steady State Queue Size Probabilities

Let the steady-state queue size probabilities during the server's breakdown state, busy state, vacation state during busy & idle period and idle state respectively be denoted by P_{br}, P_{busy}, P_{VB}, P_{VI} and P_I. Then these steady state queue size probabilities are derived by considering the equations (10.1) to (10.7) at z=1 and using results mentioned above.

i) $P_{br} = P_{br_0} + P_{br_1}$
where $P_{br_0} = BR^{0**\prime}(z,0,0)|_{z=1} = \alpha_0(1-f)\big(P_0^0(0) + P_0^1(0)\big)Q_R(1)E(S_0)E(R_0)$
and $P_{br_1} = BR^{1**\prime}(z,0,0)|_{z=1} = \alpha_1 f\big(P_0^0(0) + P_0^1(0)\big)Q_R(1)E(S_1)E(R_1)$

ii) $P_{busy} = P_{busy_0} + P_{busy_1}$
where $P_{busy_0} = P^{0*}(z,0)|_{z=1} = (1-f)\big(P_0^0(0) + P_0^1(0)\big)Q_R(1)E(S_0)$
and $P_{busy_1} = P^{1*}(z,0)|_{z=1} = f\big(P_0^0(0) + P_0^1(0)\big)Q_R(1)E(S_1)$

iii) $P_{VB} = QB^*(z,0)|_{z=1} = p(1-f)\big(P_0^0(0) + P_0^1(0)\big)\big(Q_R(1) - 1\big)E(VB)$

iv) $P_{VI} = QI^*(z,0)|_{z=1} = (1-f)\big(P_0^0(0) + P_0^1(0)\big)\sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\, E(VI)$

v) $P_I = PI^*(z,0)|_{z=1} + PI_0 = (1-f)\big(P_0^0(0) + P_0^1(0)\big)\left(Y_R(1)\frac{1-A^*(\lambda)}{\lambda} + \frac{\varphi}{\lambda}\right)$

$\big(P_0^0(0) + P_0^1(0)\big)$ can be found by using the normalizing condition $PIF_R(1) = 1$ and is given by: $P_0^0(0) + P_0^1(0) = \frac{\lambda E(X)}{(1-f)Q_R(1)}$.

Mean queue size

In this section, the mean queue sizes for the proposed model when the system is under different states are calculated. Let L_{br}, L_{busy}, L_{VB}, L_{VI} and L_I represent the expected queue size during the server's breakdown state, busy state, vacation state during busy & idle period and idle state respectively. These lengths are obtained from the derivatives of the equations (10.1) to (10.7) at $z = 1$ in which the above results are employed.

i) $L_{br} = L_{br_0} + L_{br_1}$

where $L_{br_0} = \frac{d}{dz}BR^{0**\prime}(z,0,0)|_{z=1}$
$= \alpha_0(1-f)\big(P_0^0(0) + P_0^1(0)\big)[Q_R'(1)E(S_0)E(R_0) + \frac{\lambda}{2}E(X)Q_R(1)(E(S_0)E(R_0^2) + E(S_0^2)\big(1+\alpha_0 E(R_0)\big)E(R_0))]$

and $L_{br_1} = \frac{d}{dz}BR^{1**\prime}(z,0,0)|_{z=1}$
$= \alpha_1 f\big(P_0^0(0) + P_0^1(0)\big)[Q_R'(1)E(S_1)E(R_1) + \frac{\lambda}{2}E(X)Q_R(1)(E(S_1)E(R_1^2) + E(S_1^2)\big(1+\alpha_1 E(R_1)\big)E(R_1))$
$+\lambda E(X)Q_R(1)\left(E(H_0) + \frac{f}{1-f}E(H_1)\right)E(S_1)E(R_1)]$

ii) $L_{busy} = L_{busy_0} + L_{busy_1}$

where $L_{busy_0} = \frac{d}{dz} P^{0*}(z,0)|_{z=1}$
$= (1-f)\left(P_0^0(0) + P_0^1(0)\right)[Q_R'(1)E(S_0) + \frac{\lambda}{2}E(X)Q_R(1)E(S_0^2)\left(1+\alpha_0 E(R_0)\right)]$
and $L_{busy_1} = \frac{d}{dz} P^{1*}(z,0)|_{z=1}$
$= f\left(P_0^0(0) + P_0^1(0)\right)[Q_R'(1)E(S_1) + \frac{\lambda}{2}E(X)Q_R(1)E(S_1^2)\left(1+\alpha_1 E(R_1)\right) + \lambda E(X)Q_R(1)\left(E(H_0) + \frac{f}{1-f}E(H_1)\right)E(S_1)]$

iii) $L_{VB} = \frac{d}{dz} QB^*(z,0)|_{z=1}$
$= p(1-f)\left(P_0^0(0) + P_0^1(0)\right)[Q_R'(1)E(VB) + \frac{\lambda}{2}E(X)(Q_R(1)-1)E(VB^2) + \lambda E(X)Q_R(1)\left(E(H_0) + \frac{f}{1-f}E(H_1)\right)E(VB)]$

iv) $L_{VI} = \frac{d}{dz} QI^*(z,0)|_{z=1}$
$= \frac{\lambda}{2}E(X)(1-f)\left(P_0^0(0) + P_0^1(0)\right)\sum_{k=0}^{\infty}\beta_0^k \prod_{i=0}^{k}\delta_i\, E(VI^2)$

v) $L_I = \frac{d}{dz} PI^*(z,0)|_{z=1}$
$= (1-f)\left(P_0^0(0) + P_0^1(0)\right)Y_R'(1)\frac{1-A^*(\lambda)}{\lambda}$

The total mean queue length of the system is given by
$L = L_{br_0} + L_{br_1} + L_{busy_0} + L_{busy_1} + L_{VB} + L_{VI} + L_I$

$$L = (1-f)\left(P_0^0(0) + P_0^1(0)\right)\frac{IVR(1)}{1-\rho}\left[\frac{IVR'(1)}{IVR(1)} + \frac{\lambda E(X(X-1))E(HBV_R) + (\lambda E(X))^2 E(HBV_R^2)}{2(1-\rho)}\right] \quad (19)$$

It is verified that the mean queue length of the model obtained directly from $\frac{d}{dz}PIF_R(z)|_{z=1}$ also coincides with equation (19).

IV. PARTICULAR CASES

In this section, the decomposition property for vacation queueing models is justified and some particular cases are derived.

Decomposition Property

The total probability generating function in (18) can be rewritten as

$$PIF_R(z) = \frac{(z-1)(1-\rho_{br})}{\left(z - \frac{(1-f)S_0^*(h_{\alpha_0}(w_x(z)))}{1-fS_1^*(h_{\alpha_1}(w_x(z)))}\right)}\frac{P_{idle}(z)}{P_{idle}(1)} = \frac{(z-1)(1-\rho)}{z - HBV^*(w_x(z))}\frac{IVR(z)}{IVR(1)} \quad (20)$$

where $\frac{P_{idle}(z)}{P_{idle}(1)} = \frac{\lambda E(X)}{Q_R(1)}\frac{\left(\frac{(1-f)S_0^*(h_{\alpha_0}(w_x(z)))}{1-fS_1^*(h_{\alpha_1}(w_x(z)))} - z\right)}{1-\rho_{br}}\frac{Q_R(z)}{w_x(z)}$; $\rho_{br} = \lambda E(X)\left(E(H_0) + \frac{f}{1-f}E(H_1)\right)$

The equation (20) shows that the probability generating function of the queue size probabilities of the system can be decomposed into the product of two probability generating functions, namely the PGF of number of customers in the unreliable $M^X/G/1$ retrial queue with feedback services but without server vacation and the conditional queue size distribution $\frac{P_{idle}(z)}{P_{idle}(1)}$ during the server idle period.

Total probability generating function for different vacation models

The total probability generating function of the queue size for the unreliable $M^X/G/1$ retrial feedback queueing system corresponding to different idle vacation policies is derived from the proposed model. These probability generating functions are obtained by calculating IVR(z) in (20) through the selection of δ_j's.

The selection of δ_j's and corresponding IVR(z) for different vacation policies is given below.

(i) Single Vacation Model ($\delta_0 = 1, \delta_j = 0\ \forall\ j \geq 1$):

$IVR(z) = A^*(\lambda)\frac{\beta_0}{\lambda} + M_1(z)\left[\frac{1-VI^*(w_x(z))}{w_x(z)} + p\frac{VB^*(w_x(z))-1}{w_x(z)}\right]$

(ii) Multiple Vacation Model ($\delta_j = 1\ \forall\ j \geq 0$):

$IVR(z) = M_1(z)\left[\frac{1}{1-\beta_0}\frac{1-VI^*(w_x(z))}{w_x(z)} + p\frac{VB^*(w_x(z))-1}{w_x(z)}\right]$

(iii) J-Vacation Model ($\delta_0 = 1, \delta_j = \bar{p}\ \forall\ 1 \leq j \leq J-1, \delta_j = 0\ \forall\ j \geq J$):

$IVR(z) = \frac{A^*(\lambda)}{\lambda}\left((1-\bar{p})\beta_0\frac{1-(\beta_0\bar{p})^{J-1}}{1-\beta_0\bar{p}} + \beta_0^J\bar{p}^{J-1}\right) + M_1(z)\left[\frac{1-(\beta_0\bar{p})^J}{1-\beta_0\bar{p}}\frac{1-VI^*(w_x(z))}{w_x(z)} + p\frac{VB^*(w_x(z))-1}{w_x(z)}\right]$

(iv) Non-Vacation Model ($\delta_j = 0\ \forall\ j \geq 0, p = 0$):

$IVR(z) = \frac{A^*(\lambda)}{\lambda}$

Classical unreliable $M^X/G/1$ queueing model with vacation and infinite feedback

The total probability generating function of the queue size probabilities of the retrial queueing model $PIF_R(z)$ given in (18) can be reduced to the corresponding classical model under the condition $A^*(\lambda) \to 1$.

i.e. $PIF_R(z) = (1-f)\left(P_0^0(0) + P_0^1(0)\right)\frac{z-1}{z - \frac{(1-f)S_0^*(h_{\alpha_0}(w_x(z)))}{1-fS_1^*(h_{\alpha_1}(w_x(z)))}\left(1-p+pVB^*(w_x(z))\right)}\left(\frac{\varphi}{\lambda} + \sum_{k=0}^{\infty}\beta_0^k\prod_{i=0}^{k}\delta_i\frac{1-VI^*(w_x(z))}{w_x(z)} + p\frac{VB^*(w_x(z))-1}{w_x(z)}\right)$

V. NUMERICAL ANALYSIS

In this section, the queue size probabilities and mean queue lengths corresponding to the parameters of different distributions are analysed. The distribution of each random variable and their measures used for numerical computation of the model are listed in the following table.

Random Variable (Y)	**Distribution**	**Mean E(Y)**	**Second Order Moments E(Y^2)**
Retrial Time (A)	Exponential (ν_1) $\nu_1 = 9$	$\frac{1}{\nu_1}$	$\frac{2}{\nu_1^2}$
Vacation Time during Idle state (VI)	Erlang (c, η_1) $c = 4, \eta_1 = 0.3$	$\frac{1}{\eta_1}$	$\frac{c+1}{c(\eta_1^2)}$
Vacation Time during Busy period (VB)	Gamma (c_1, η) $c_1 = 2, \eta = 5$	$\frac{c_1}{\eta}$	$\frac{c_1(c_1+1)}{\eta^2}$
Primary Service Time (S_0)	Erlang (k, μ_0) $k = 5, \mu_0 = 0.8$	$\frac{1}{\mu_0}$	$\frac{k+1}{k(\mu_0^2)}$
Feedback Service Time (S_1)	Erlang (k, μ_1) $k = 5, \mu_1 = 0.64$	$\frac{1}{\mu_1}$	$\frac{k+1}{k(\mu_1^2)}$
Batch Size (X)	Geometric (p_1) $p_1 = 0.45$	$\frac{1}{1-p_1}$	$\frac{2p_1}{(1-p_1)^2}$
Repair Time during primary service (R_0)	Exponential (r_0) $r_0 = 0.6$	$\frac{1}{r_0}$	$\frac{2}{r_0^2}$
Repair Time during feedback service (R_1)	Exponential (r_1) $r_1 = 0.36$	$\frac{1}{r_1}$	$\frac{2}{r_1^2}$

Fig.1 & Fig.2 depict the effect of batch arrival rate (λ) on the queue size probabilities and the mean queue lengths at different states of the system. From the computations, it is observed that as λ increases,

i) queue size probabilities when the system is busy (P_{busy}), under repair (P_{br}) and in busy vacation state (P_{VB}) gradually increase, whereas the queue size probability during the idle vacation of the server (P_{VI}) and that at the idle state of the system (P_I) decrease and

ii) total mean queue length of the system (L) along with the expected queue lengths during the busy state (L_{busy}), breakdown state (L_{br}), busy vacation state (L_{VB}) and idle state (L_I) increase but those mean queue lengths when the server is in idle vacation (L_{VI}) decreases.

Figure 1

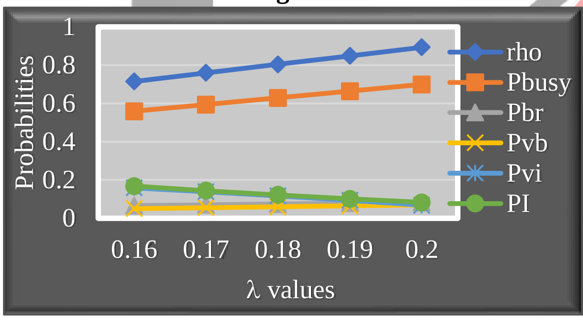


Figure 2

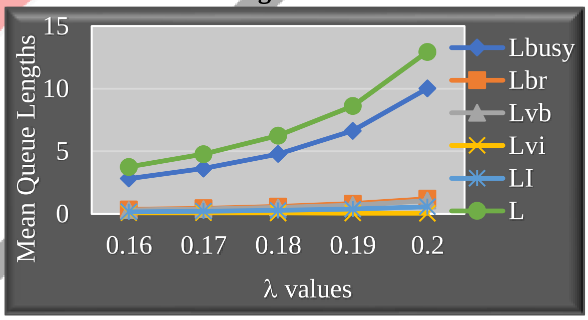


Fig.3 shows that the change in the feedback probability (f) has the similar impact on the queue size probabilities that of the batch arrival rate (λ).

Figure 3

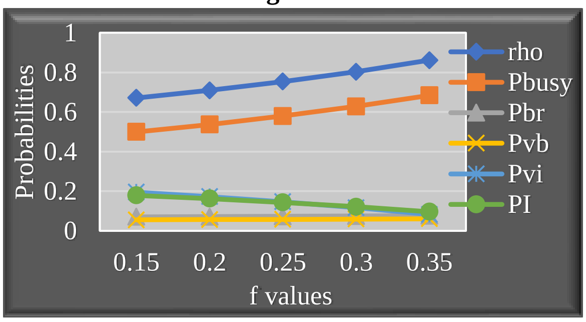


Fig.4 & Fig.5 show that the effects of regular and feedback service rates μ_0 and μ_1 and breakdown rates α_0 and α_1 on the total mean queue length (L). It is observed that

i) As μ_0 or μ_1 decreases, both P_{busy} and L decrease and
ii) P_{br} and L increase with the increase in α_0 or α_1.

Figure 4

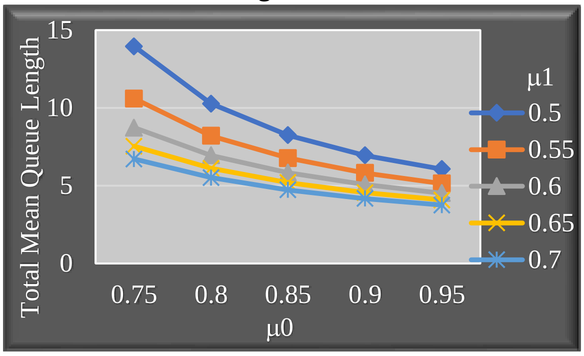
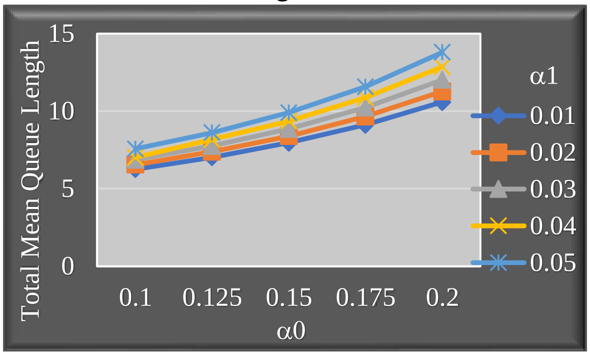


Figure 5



VI. CONCLUSION

The present study shows that the queueing models need not be treated separately for each vacation policy. Instead, if the systems are considered under Multiple Adapted Vacation policy, then the results for every classical vacation queueing models including the non-vacation case can be deduced. The stochastic decomposition property for vacation queues is established and a sample of the effects of system parameters on performance measures is discussed numerically.

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