Fuzzy join hyperlattice

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***Abstract*: Hyperlattices are the most developing area in the Lattice Theory. In this paper we introduce notion of Fuzzy Theory in Join hyperlattices and also we investigate the properties of Homomorphism of Fuzzy Join hyperlattices.**

***Keywords*: Hyperlattice, join hyperlattice, fuzzy hyperlattice, homomorphism**

Introduction:
Lattice theory is one of the most important and vast area in mathematics. The notion of hypergroup was also introduced in the lattice theory. Now, the recent trends in lattice theory are fuzzy lattices and fuzzy hyperlattices. Now we introduce fuzzy theory in join hyperlattice and we investigate some of the properties of them. We also derive the relation between the fuzzy join hyperlattice and join hyperlattice. Also, we investigate the properties of homomorphism of fuzzy join hyperlattices.

1. PRELIMINARIES

In this section we see some of the basic definitions and conditions, which we use in this paper.

Definition 1.1:

An Algebra (L, ∨, ∧) is called a Lattice[4], if L is a non-empty set, ∧ and ∨ are binary operations on L, then both ∨ and ∧ are

1) Idempotent
2) Commutative, and
3) Associative, and they satisfy the
4) Absorption law.

Proposition 1.2:

Let L be a non-empty set with two binary operations ∧ and∨. Let a, b, c ∈ L, then the following conditions are satisfied:

1) $a \wedge a = a$ and $a \vee a = a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
4) $(a \wedge b) \vee a = a$ and $(a \vee b) \wedge a = a$

Then, (L,∨,∧) is a Lattice.

Definition 1.3:

Let L be a non-empty set with two hyper operations ∧ and∨. The triplet (L,∨, ∧) is called as hyperlattice if the following identities holds for all a, b, c ∈ L.

1) $a \in a \wedge a$ and $a \in a \vee a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
4) $a \in a \wedge (a \vee b)$ and $a \in a \vee (a \wedge b)$

Definition 1.4:

Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two hyperlattices. A map f: $(L_1 \longrightarrow L_2)$ is said to be a

1) weak homomorphism if $f(a \wedge_1 b) \subseteq f(a) \wedge_2 f(b)$ and $f(a \vee_1 b) \subseteq f(a) \vee_2 f(b)$ for all a, b ∈ L_1.
2) Homomorphism if $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$ and $f(a \vee_1 b) = f(a) \vee_2 f(b)$ for all a, b ∈ L_1.

If such a homomorphism f is surjective, then f is called an epimorphism.

If the homomorphism f is injective, then f is called a monomorphism

If f is bijective, then f is called as isomorphism from $(L_1, \vee_1, \wedge_1)$ to the hyperlattice $(L_2, \vee_2, \wedge_2)$.

Note 1.5:

Let L be a non-empty set and F*(L) be the set of all non-zero fuzzy subset of L. A map ∘ is called as a fuzzy hyperoperation on L defined by ∘ : L × L ⟶ F*(L), where any pair (a, b) of elements of L × L is associated with a non-zero fuzzy subset a ∘ b.

If a and b are two non-zero fuzzy subsets of a hypergroupoid (L, ∘) for all x, y ∈ L, then we define

1) (x ∘ a) (y) = $sup_{t \in L}$ { (x ∘ t) (y) ∩ a (t) }, (a ∘ x)(y) = $sup_{t \in L}$ {a (t) ∩ (t ∘ x) (y)}
2) (a ∘ b) (x) = $sup_{t \in L}$ { a (p) ∩ (p ∘ q) (y) ∩ b (q) }

Remark 1.6:

If A is a non-empty subset of L, then we denote the characteristic function of A by χ_A, where for all y ∈ L, we have

$$\chi_A(y) = \begin{cases} 1, & y \in A \\ 0, & y \notin A \end{cases}$$

In particular, for all x, y ∈ L, if A = {x}, then we denote $\chi_{\{x\}}$(y) = χ_b(y), which means that

$$\chi_x(y) = \begin{cases} 1, & y = a \\ 0, & y \neq a \end{cases}$$

Definition 1.7:

Let L be a non-empty set with two hyperoperation ∨ and ∧. The Triplet (L,∨,∧) is called a fuzzy hyperlattice[1], if the following identities holds for all a, b, c ∈ L.

1) (a ∨ a) (a) > 0 and (a ∧ a)(a) > 0
2) a ∨ b = b ∨ a and a ∧ b = b ∧ a
3) (a ∨ b) ∨ c = a ∨ (b ∨ c) and (a ∧ b) ∧ c = a ∧ (b ∧ c)
4) (a ∨ (a ∧ b)) (a) > 0 and (a ∧ (a ∨ b)) (a) > 0.

II. RELATION BETWEEN FUZZY JOIN HYPERLATTICE AND JOIN HYPERLATTICE

In this chapter, we investigate the relation between fuzzy join hyperlattice and join hyperlattice [2]. We derive prove some of the theorems on them.

Definition 2.1:

Let L be a non-empty set,∨: L × L ⟶ F*(L), be a fuzzy hyperoperation and ∧ : L × L ⟶ L be a operation. Then, (L, ∨, ∧) is a fuzzy join hyperlattice if for all x, y, z ∈ L the following conditions holds:

1) (x ∨ x) (x) >0 and (x ∧ x) (x) > o
2) x ∨ y = y ∨ x and x ∧ y = y ∧ x
3) (x ∨ y) ∨ z = x ∨ (y ∨ z) and (x ∧ y) ∧ z = x ∧ (y ∧ z)
4) (x ∨ (x ∧ y)) (x) > 0 and (x ∧ (x ∨ y)) (x) > 0.

Example 2.2:

Let (L, ∩, ∪) be a lattice. If we define the fuzzy hyperoperation on L for all x, y ∈ L,

x ∨ y = $\chi_{\{x,y\}}$ and the operation

x ∧ y = $\chi_{x \cap y}$, then (L, ∨, ∧) is a fuzzy join hyperlattice.

Proof:

1) It is clear that
(x ∨ x) (x) >0 and (x ∧ x) (x) > o and
2) x ∨ y = y ∨ x and x ∧ y = y ∧ x [since x, y ∈ L]

3) Fuzzy associative laws have to be proved, for all x, y, z, a ∈ L

$((x \vee y) \vee z)(a) = sup_{t\in L} \{(x \vee y)(t) \cap (t \vee z)(a)\}$
$= sup_{t\in L} \{ \chi_{\{x,y\}}(t) \cap (t \vee z)(a)\}$
$= (x \vee z)(a) \cup (y \vee z)(a)$
$= \chi_{\{x,z\}}(a) \cup \chi_{\{y,z\}}(a)$
$= \chi_{\{x,y,z\}}(a)$

and

$x \vee (y \vee z)(a) = sup_{t\in L} \{(x \vee t)(a) \cap (y \vee z)(t)\}$
$= sup_{t\in L} \{(x \vee t)(a) \cap \chi_{\{y,z\}}(t)\}$
$= (x \vee y)(a) \cup (x \vee z)(a)$
$= \chi_{\{x,y\}}(a) \cup \chi_{\{x,z\}}(a)$
$= \chi_{\{x,y,z\}}(a)$

Hence $(x \vee y) \vee z = x \vee (y \vee z)$.
Similarly, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ can be proved.

4) Fuzzy absorption law:
for all x, y, z ∈ L and a ∈ L,

$(x \vee (x \wedge y))(x) = sup_{t\in L} \{(x \vee t)(x) \cap (x \wedge y)(t)\}$
$= sup_{t\in L} \{(x \vee t)(x) \cap \chi_{\{x\cap y\}}(t)\}$
$= (x \vee (x \cap y))(x)$
$= \chi_{\{x,x\cap y\}}(x) = 1 > 0$

$(x \wedge (x \vee y))(x) = sup_{t\in L} \{(x \wedge t)(x) \cap (x \vee y)(t)\}$
$= sup_{t\in L} \{(x \wedge t)(x) \cap \chi_{\{x,y\}}(t)\}$
$= (x \wedge x)(x) \cup (x \wedge y)(x)$
$\geq (x \wedge x)(x) > 0$

The proof is complete.

Example 2.3:

Let (L, ∩, ∪) be a lattice. If we define the fuzzy hyperoperation ∨ on L for all x, y ∈ L, as

$x \vee y = \chi_{\{x,y\}}$ and the operation ∧ is defined as for all a ∈ L,

$$(x \wedge y)(a) = \begin{cases} \{½\}, & a = (x \cap y) \\ 0, & \text{otherwise} \end{cases}$$

Then (L, ∨, ∧) is a fuzzy join hyperlattice.

Example 2.4:

Let (L,∩, ∪) be a lattice. If we define the fuzzy hyperoperation ∨ on L for all x, y ∈ L, as

$x \vee y = \chi_{\{x\cup y\}}$ and the operation ∧ is defined as for all a ∈ L,

$$(x \wedge y)(a) = \begin{cases} \{½\}, & a = (x \cap y) \\ 0, & \text{otherwise} \end{cases}$$

Then (L, ∨, ∧) is a fuzzy join hyperlattice.

Definition 2.5:

Let L be a non-empty set,∨: L × L → F* (L) be a hyper operation and ∧ : L × L →L be an operation. Then (L,∨, ∧) is a join hyperlattice[3] if for all x, y, z Є L. The following conditions are satisfied:

1) $x \in x \vee x$ and $x = x \wedge x$
2) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
3) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
4) $x \in x \wedge (x \vee y) \cap x \vee (x \wedge y)$

On considering a fuzzy join hyperlattice (L,∨, ∧) and defining the hyperoperations on L for all x, y Є L,

$x \otimes y = \{a \in L \mid (x \vee y)(a) > 0\}$ and

$x \oplus y = \{a \in L \mid (x \wedge y)(a) > 0\}$, then we obtain a hyperlattice.

Theorem 2.6:

If $(L, \vee, \wedge)$ is a fuzzy join hyperlattice, then $(L, \otimes, \oplus)$ is a join hyperlattice, which is called the associated join hyperlattice.

Proof:

1) clearly, $x \in x \otimes x$ and $x \in x \oplus x$
2) $x \otimes y = y \otimes x$ and $x \oplus y = y \oplus x$
3) Associative laws:

We claim that $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, for all $x, y, z \in L$, then for all $a \in L$,
if $a \in x \otimes (y \otimes z)$,

then there exists $b \in y \otimes z$, such that $a \in x \otimes b$ which gives $(y \vee z)(b) > 0$ and $(x \vee b) > 0$

Hence, $x \vee (y \vee z)(a) = sup_{t \in L} \{(x \vee t)(a) \cap (y \vee z)(t)\}$

$\geq (x \vee b)(a) \cap (y \vee z)(b) > 0$

Since $x \vee (y \vee z) = (x \vee y) \vee z$, it follows that $((x \vee y) \vee z)(a) > 0$

which implies that

$sup_{t \in L} \{(x \vee y)(t) \cap (t \vee z)(a)\} > 0$

Hence there is $t' \in L$, such that

$(x \vee y)(t') > 0$ and $(t' \vee z)(a) > 0$.

This shows that $t' \in x \otimes y$ and $a \in t' \otimes z$.

Therefore, we get $a \in (x \otimes y) \otimes z$

Hence, $x \otimes (y \otimes z) \subseteq (x \otimes y) \otimes z$.

Conversely, $(x \otimes y) \otimes z \subseteq x \otimes (y \otimes z)$.

So, we arrive at the equality,

$x \otimes (y \otimes z) = (x \otimes y) \otimes z$.

Similarly, we can show that,

$x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

4) Absorption Law:

$(x \vee (x \wedge y))(x) = sup_{t \in L} \{(x \vee t)(x) \cap (x \wedge y)(t)\} > 0$, then there exists $t' \in L$ such that,

$(x \vee y)(t') > 0$ and $(x \vee t')(x) > 0$ which shows that there exists $t' \in x \oplus y$,

such that $x \in x \otimes t'$.

Therefore, $x \in x \otimes (x \oplus y)$.

Similarly, we get $x \in x \oplus (x \otimes y)$.

Combining 1), 2), 3) and 4) we have that $(L, \otimes, \oplus)$ is a join hyperlattice.

Remark 2.7:

Now, on considering a join hyperlattice $(L, \otimes, \oplus)$, and defining the hyperoperation on L: for all $x, y \in L$,

$x \vee y = \chi_{x \otimes y}$

x ∧ y = $\chi_{x\oplus y}$, then we obtain a fuzzy join hyperlattice.

Throrem2.8:

If (L, ⊗, ⊕), is a join hyperlattice, then (L, ∨, ∧) is a fuzzy join hyperlattice, which is called the associated fuzzy join hyperlattice.

Proof:

1) (x ∨x) (x) > 0 and (x ∧ x)(x) > 0 is clear and

2) x ∨y = y ∨x and x ∧ y = y ∧ x also holds

3) Fuzzy associative law:

For all x, y, z, a Є L, we have

$$((x \vee y) \vee z)(a) = sup_{t\in L} \{(x \vee y)(t) \cap (t \vee z)(a)\}$$
$$= sup_{t\in L} \{ \chi_{x\otimes y}(t) \cap \chi_{t\otimes z}(a)\}$$
$$= \begin{cases} 1, \text{ if } a \in (x \otimes y) \otimes z \\ 0, \text{ otherwise} \end{cases}$$

Then,

$$(x \vee (y \vee z))(a) = sup_{t\in L} \{(x \vee t)(a) \cap (y \vee z)(t)\}$$
$$= sup_{t\in L} \{ \chi_{x\otimes t}(a) \cap \chi_{y\otimes c}(t)\}$$
$$= \begin{cases} 1, \text{ if } a \in (x \otimes y) \otimes z \\ 0, \text{ otherwise} \end{cases}$$

We get,

(x ∨y) ∨z = x ∨ **(** y ∨z) [since, x ⊗ (y ⊗ z) = (x ⊗y) ⊗ z]

Similarly, we can prove that

(x ∧ y) ∧ z = x ∧ (y ∧ z)

4) Fuzzy Absorption law,

For all x, y Є L we have, $(x \vee (x \wedge y))(x) = sup_{t\in L} \{(x \vee t)(x) \cap (x \wedge y)(t)\}$

$$= sup_{t\in L} \{ \chi_{x\otimes t}(x) \cap \chi_{x\otimes y}(t)\}$$
$$= \chi_{x\otimes(x\oplus y)}(x)$$
$$= 1 > 0$$

Hence the proof is completed.

From the above two theorems we have the following facts.

Denote the class of all fuzzy join hyperlattice by fuzzy join hyper lattice and the class of all join hyperlattice by join hyper lattice. We can form two maps

1) θ: FJHL → JHL, θ ((L, ∨, ∧)) = (L, ⊗, ⊕), where for any x, y Є L, we have

x ⊗ y = {a Є L | (x ∨y) (a) > 0} and

2) π: JHL → FJHL, π((L, ⊗, ⊕)) = (L, ∨, ∧), where for any x, y Є L, we have

(x ∨y) = $\chi_{\{x\otimes y\}}$ and (x ∨y) = $\chi_{\{x\oplus y\}}$

III. HOMOMORPHISM OF FUZZY JOIN HYPERLATTICE AND JOIN HYPERLATTICE

In this section we introduce the notion of homomorphism of fuzzy join hyperlattices and we study the connections between join hyperlattices homomorphism and join hyperlattice homomorphism.

Definition 3.1:

Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two fuzzy join hyperlattices. A map f: $(L_1 \longrightarrow L_2)$ is said to be

1) Weak homomorphism if f (x $\vee_1$ y) $\subseteq$ f(x) $\vee_2$, f(y) and f(x $\wedge_1$ y) $\subseteq$ f(a) $\wedge_2$ f(b) for all x, y $\in L_1$.
2) Homomorphism if f (x $\vee_1$ y) = f(x) $\vee_2$ f(y) and f(x $\wedge_1$y) = f(a) $\wedge_2$ f(b)
for all x, y $\in L_1$.

If such a homomorphism is said to be surjective, injective or bijective then the mapping f is called as epimorphism, monomorphism or isomorphism from the join hyperlattice $(L_1,\vee_1,\wedge_1)$ to the join hyperlattice $(L_2, \vee_2, \wedge_2)$ respectively.

Theorem 3.2:

Let $(L_1,\vee_1, \wedge_1)$ and $(L_2, \vee_2,\wedge_2)$ be two fuzzy join hyperlattices and

$(L_1, \otimes_1, \oplus_1) = \theta((L_1,\vee_1, \wedge_1))$ and $(L_2,\otimes_2,\oplus_2) = \theta((L_2, \vee_2,\wedge_2))$ be their associated join hyperlattices , respectively. If f: $L_1 \rightarrow L_2$ is a weak homomorphism of fuzzy join hyperlattices, then f is a weak homomorphism of the associated join hyperlattices, too.

Proof:

Since f: $L_1 \rightarrow L_2$ is a weak homomorphism of fuzzy join hyperlattices, we have

f (x$\vee_1$y) $\subseteq$ f(x) $\vee_2$f(y)and f(x$\wedge_1$y)) $\subseteq$ f(x) $\wedge_2$f(y) for all x, yϵL_1.

This shows that

(f (x$\vee_1$y))(b) $\leq$ (f(x) $\vee_2$f(y))(b) and

(f (x$\wedge_1$y))(b) $\leq$ (f(x) $\wedge_2$f(y))(b) for all bϵL_2.

Let a ϵ x $\otimes_1$y, which means that (x$\vee_1$y ((a) > 0 and let b = f (a)

Then, (f(x$\vee_1$y)) (b) = sup{(x$\vee_1$y) (t) | f (t) = b, t ϵL_1}

$\geq$(x$\vee_1$y) (a) >0

It shows that (f(x) $\vee_2$f(y) (b)>0.

Hence b = f (a) ϵ f(x) $\otimes_2$ f(y).

Therefore we get, f(x $\otimes_1$ y) $\subseteq$ f(x) $\otimes_2$ f(y).

Similarly, we can show that

f(x $\oplus_1$ y) $\subseteq$ f(x) $\oplus_2$f(y).

Therefore, f is a weak homomorphism between the associated join hyperlattices $(L_1,\otimes_1, \oplus_1)$ and $(L_2, \otimes_2, \oplus_2)$.

IV.CONCLUSION

Hence, we have successfully introduced the fuzzy join hyperlattice. And we investigated some of their properties.

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