A Study on Matrices in Elimination Theory

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***Abstract*: In this paper, we will study systematic methods for eliminating variable from system of polynomial equation. The task for which elimination theory was conceived is to find the complete solution of a system of algebraic equations. The resultant methods is a major powerful tool for variable elimination polynomial system solving. The main strategy of this elimination theory will be given in examples.**

1. INTRODUCTION:

Many problems in linear algebra and other branches of science to solving a system of linear equations in a number of variables. This in turn means finding common solution to some polynomial equation of degree one. We are faced with non-linear system of polynomial equation in more than one variable. Elimination theory is most important for both algorithmic and complexity aspect of polynomial system solving. It also impacts several other areas of mathematics like numerical analysis, complexity, linear algebra etc. It is general about eliminating a number of unknowns from a system of polynomial equations in one (or) more variables to get an equivalent system. The importance of elimination theory, let us start by considering the following example.

Definition 2.1: Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_1 x + a_0$, $g(x) = b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x + b_0$ be two polynomials of degree *m* and *n* respectively such that $a_m \neq 0$ or $b_n \neq 0$. If $m \leq n$, we define the resultant of *f(x)* and *g(x)* to be following determinant

$$\text{Res}(f(x), g(x)) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_1 & b_0 \end{vmatrix}$$

We Notice that Res (*f(x), g(x)*) is the determinate of a square matrix of size $m + n$.

THEOREM 2.2:

Let $f(x)=a_m x^m+a_{m-1}x^{m-1}+....+a_1x+a_0$, $g(x)=b_nx^n+ b_{n-1}x^{n-1}+......+b_1x+b_0$, be two polynomials of degree *m* and *n* respectively such that $a_m \neq 0$ or $b_n \neq 0$ then the system F(x)=0,G(x)=0. Has a solution if and only if Res(f(x),g(x))=0

Example 2.3: Without solving the polynomial equation, show that the following system $X^3-3x^2+5x-3=0$; $2x^2-7x+5=0$ has solutions.

SOLUTION:

We compute the resultant of two polynomials $f(x) = x^3-3x^2+5x-3$, $g(x)=2x^2-7x+5$

$$\text{Res}(f(x), g(x)) = \begin{vmatrix} 2 & -7 & 5 & 0 & 0 \\ 0 & 2 & -7 & 5 & 0 \\ 0 & 0 & 2 & -7 & 5 \\ 1 & -3 & 5 & -3 & 0 \\ 0 & 1 & -3 & 5 & -3 \end{vmatrix}$$

Therefore the polynomials f(x), g(x) have a common root by the about theorem

Now we can use theorem to determine if a polynomial system in more than one variable has a solution,

The polynomials in the systems as polynomials in one variable with coefficients polynomials in the other variables.

Example 2.4: Solve the following system $9x^2+4y^2-18x+16y-11=0$, $x^2+y^2-9=0$

SOLUTION:

We may look at this system as polynomials in y with coefficients polynomials in x:

$4y^2+16y+(9x^2-18x-11)=0$

$Y^2+(x^2-9)=0$

In order to have a common solution, one must have :

$$\begin{vmatrix} 4 & 16 & 9x2-18x-11 & 0 \\ 0 & 4 & 16 & 9x2+8x-11 \\ 1 & 0 & x2-9 & 0 \\ 0 & 1 & 0 & x2-9 \end{vmatrix} =0$$

This is equivalent to

$25x^4-180x^3+574x^2-900x+625=0$

This example is reduced then to solving a polynomial and in one variable x. Since the solution of this equation does not look easy, one can use a numerical approach to estimate the solutions .this system can be written as

$(x-1)^2/4+(y+2)^2/9=1$

$(x-0)^2+(y-0)^2=9$

So any solution to the system is an intersection of an ellipse and a circle that can be found geometrically

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