Observations on the ternary cubic Diophantine equation $x^2 + y^2 - xy = 52z^3$

[1]Dr. R. Anbuselvi, [2]R. Nandhini

[1]Associate Professor, [2]Lecturer
Department of Mathematics
[1]A.D.M College for Women (Autonomous), Nagapattinam, India
BDU Model College, Thiruthuraipoondi, India

***Abstract*: The ternary cubic Diophantine equation is analyzed for its infinitely many non-zero distinct integral solutions. A few interesting properties among the solutions are presented.**

***Keywords*: Ternary cubic equation, integral solutions**

1. Introduction

Integral solutions for the cubic homogeneous or non -homogeneous diophantine equations is an interesting concept, as it can be seen from [1-3]. In [4-13] a few special cases of ternary cubic diophantine equations are studied. In this communication, we present the integral solutions of yet another ternary cubic equation $x^2 + y^2 - xy = 52z^3$. A few interesting relations between the solutions are obtained.

Notations

P_n^m	=	Pyramidal number of rank n with size m
SO_n	=	Stella Octangular number of rank 'n'
Obl_n	=	Oblong number of rank 'n'
OH_n	=	Octahedral number of rank 'n'
Tha_n	=	Thabit-ibn-Kurrah number
$Carol_n$	=	Carol number
M_n	=	Mersenne number
Ky_n	=	Kynea number
$t_{m,n}$	=	Polygonal number of rank 'n' with size 'm'

2. Method of Analysis

The cubic equation under consideration is

$$x^2+y^2-xy = 52z^3 \quad (1)$$

Assuming $x = u + v$, $y = u - v$, $u \neq v$ (2)

in (1), it is written as

$$u^2 + 3v^2 = 52z^3 \quad (3)$$

Here, we rewrite 52 in five different ways and hence obtain different patterns of solutions to (1) which are illustrate as follows:

Pattern 1

Assume $z = z(a, b) = a^2 + 3b^2$ $a, b \neq 0$ (4)

Write 52 as

$$52=(7+i\sqrt{3})(7-i\sqrt{3}) \quad (5)$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(a+i\sqrt{3}b)^3(7+i\sqrt{3}) \quad (6)$$

On comparing real and imaginary parts on both sides, we get

$$u = u(a, b) = 7a^3 - 9a^2b - 63ab^2 + 9b^3$$
$$v = v(a, b) = a^3 + 21a^2b - 9ab^2 - 21b^3$$

Substituting the values of u and v in (2), we get

$x = x(a, b) = 8a^3+12a^2b - 72ab^2-12b^3$ (7)
$y = y(a, b) = 6a^3-30a^2b - 54ab^2+30b^3$ (8)

Thus (4), (7) and (8) represent the non-zero distinct integer solutions of (1)

Properties

1) $x(a,1) - y(a,1) - 4P_a^4 - t_{102,a} + t_{22,a} \equiv -20 \pmod{22}$
2) $y(a,1) + z(a,1) - 4SO_a - t_{38,a} + t_{12,a} \equiv -9 \pmod{55}$
3) $y(1,b) - z(1,b) - 45OH_b + t_{52,b} - t_{66,b} \equiv 5 \pmod{100}$
4) $3z(a,a+1)+x(a,a+1)-24P_{a^5}+72Pr_a+2SO_a+ t_{30,a}+ t_{22,a} \equiv -51 \pmod{114}$
5) $z(a,a+1) - t_{58,a}+ t_{50,a} \equiv 3 \pmod{10}$

Pattern 2

Instead of (5), we write 52 as

$52 = (5+i3\sqrt{3})(5-i3\sqrt{3})$ (9)

Following the procedure presented in pattern 1, we obtain the integer value of x, y, z satisfy (1) to be

$x = x(a,b) = 8a^3-12a^2b-72ab^2+12b^3$ (10)
$y = y(a,b) = 2a^3-42a^2b-18ab^2+42b^3$ (11)
$z = z(a,b) = a^2+3b^2$

Thus (4) ,(10) and (11) represent the non-zero distinct integer solutions of (1)

Properties

1) $x(a,2)+y(a,2)-3SOa - t138,a+t18,a \equiv -87 \pmod{153}$
2) $x(1,b)-6SO_b+t_{146,b} \equiv 8 \pmod{77}$
3) $z(2,b)=y(2,b)-63OH_b+t_{46,b}+t_{24,b} \equiv 20 \pmod{220}$
4) $z(a,a(a+1))-3Obl_a - t_{22,a}+ t_{14,a} \equiv 0 \pmod{12}$
5) $z(2^n,2)-13 \equiv M_{2n}$

Pattern 3

52 can also be written as

$52 = (2+i4\sqrt{3})(2 - i4\sqrt{3})$ (12)

Proceeding as in pattern 1, the non-zero distinct integral solutions of (1) are given by

$x = x(a,b) = 6a^3-30a^2b-54ab^2+30b^3$ (13)
$y = y(a,b) = -2a^3-42a^2b+18ab^2+42b^3$ (14)

Thus (4), (13) and (14) represent the non-zero distinct integer solutions of (1)

Properties

1) $z(2, b)+y(2, b) - 6SO_b - 60P_6^5 - t_{12,b} - t_{10,6} \equiv -12 \pmod{155}$
2) $y(1,b)-z(1,b)-21SO_b - t_{92,b}+ t_{62,b} \equiv -3 \pmod 6$
3) $x(a,1)-3SO_a+t_{102,a} - t_{42,a} \equiv 30 \pmod{81}$
4) $z(1,b+1) - t_{90,b} + t_{84,b} \equiv 4 \pmod 9$
5) $z((2^n+1),1)-5 = Ky_n$

Pattern 4

Apart from (5) ,(9) and (12) we write 52 as

$52 = \frac{[50+i4\sqrt{3})(50-i4\sqrt{3}]}{49}$ (15)

For this choice, the corresponding solutions of (1) are represented by

$x = x(A,B)=2646A^3+5586A^2B-23814AB^2-5586B^3$ (16)
$y = y(A,B)=2254A^3-9114A^2B-20286AB^2+9114B^3$ (17)
$z = z(A,B)=49A^2+147B^2$ (18)

Properties

1) $y(1,n)- 13671OH_n+t_{40002,n}+ t_{574,n} \equiv 2254 \pmod{33955}$
2) $y(1,b)-x(1,b)-7350SO_b-t_{9002,b}+t_{1946,b} \equiv -392 \pmod{3822}$
3) $z(1,b)-x(1,b)-8379OH_b-t_{49002,b}+t_{1080,b} \equiv -2597 \pmod{15582}$
4) $z((2^n-1),1)-245 = 49\,Carol_n$
5) $z(1,2^n)-98= 49\,Tha_{2n}$

Pattern 5

Equation (3) can be written as

$u^2+3v^2=52z^3*1$ (19)

Write '1' as

$1 = \frac{(1+i\sqrt{3})\ (1-i\sqrt{3})}{4}$ (20)

Define $(u+i\sqrt{3}v) = \frac{(7+i\sqrt{3})\ (a+i\sqrt{3}b)^3\ (1+i\sqrt{3})}{2}$ (21)

Equating real and imaginary parts, we have

$u = 2(a^3-18a^2b-9ab^2+18b^3)$

$v = 2(2a^3+3a^2b-18ab^2-3b^3)$

Since our aim is to find integer solution substituting the values of u,v in (2),we obtain the distinct nonzero integral solutions to (1) as

$x = x(a,b)=2(3a^3-15a^2b-27ab^2+15b^3)$

$y = y(a,b)=2(-a^3-21a^2b+9ab^2+21b^3)$

Properties

1. $x(1,b)+y(1,b)-36SO_b+t_{82,b}-t_{10,b} \equiv -4 \pmod{72}$
2. $x(a,a+1)+3SO_b+60P_a^5+t_{12,a}+t_{10,6} \equiv 7 \pmod 8$
3. $z(a+1,a)-t_{178,a}+t_{170,a} \equiv 1 \pmod 6$
4. $y(2^n,1)-16= 18M_n-42M_{2n}-2M_{3n}$

Conclusion

To conclude one may search for other patterns of solutions to (1) along with their properties.

References

1. Dickson LE. History of the Theory of Numbers, Diophantine Analysis, New York, Dover,2005

2. Mordell LJ. Diophantine Equations, Academic Press,New York, 1969.

3. Carmichael RD. The Theory of Numbers and Diophantine Analysis, New York, Dover, 1959

4. Gopalan MA, Anbu Selvi R. Integral solutions of Ternary Cubic Diophantine Equations $x^2 + y^2 +4N = zxy$, Pure and Applied Mathematics Sciences,2008; LXVI I.(l-2):I 07-111.

5. Gopalan MA, Manju Somanath,Vanitha N. On Ternary Cubic Diophantine Equations $x^2 + y^2 = 2z^3$,Advances in Theoretical and Applied Mathematics, 2006; I (3):227-231.

6. Gopalan MA, Manju Somanath, Vanitha N. On Ternary = Cubic Diophantine equations $x^2 + y^2 = 2z^3$, Acta Ciencia Indica,2007; XXXU IIM(3):705-707

7. Gopalan MA, Pandichel vi V. Integral solutions of Ternary Cubic Equation $x^2 - xy+ y^2 = (k^2 -2k +4) z^3$, Pacific Asian Journal of Mathemat ics. 2008:2(1-2):91-96.

8. Gopalan MA, Kaliga Rani J. Integral solutions of $x^2 - ay^2 = (a -1) z^3$ (a-1 and a are square free), Impact J. Sci. Tech.2(4):201-204.

9. Gopalan MA, Devibala S, Manju Somanath. Integral solutions of $x^3 +x+ y^3 + y =4z(z -2) (z+2)$ Impact J.Sci. Tech. 2(2):65-69.

11. Gopalan MA, Kaliga Rani. J., Integral solutions of $x^3 + y^3 + 8k (x + y) = (2k + 1) z^3$ Bulletin of Pure and Applied Sciences. 2010:29E(l):95-99.

12. Gopalan MA, Janaki G Integral solutions of $x^2 - y^2 + xy = (m^2- 5n^2) z^3$ Antartica J.Math 2010, 7(1):63-67.

13. Gopalan MA, Shanmuganandham P. On the equation $x^2 + xy - y^2 = (n^2 + 4n -1) z^3$, Bulletin of pure and applied Sciences. 2010,29F.(2) 231-235