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A STUDY ON THE HAMILTONIAN PATHS, LATIN SQUARES, AND GROUPS

Sunilkumar S. Badiger

Lecturer (Selection Grade)
Government Polytechnic, MUDHOL

***Abstract*: Ever since the most significant contributor to mathematics Leonhard Euler contributed to the study of Knight's Tour problem on a conventional chess board, several mathematicians and enthusiasts contributed to this idea. Beginning with such a Knight Tour, I found a method to convert that into Permutation Group Structure using the Amino Acid DNA base pairs (A – T) and (G – C), the so-called blueprints making up the whole Genome structure. In this paper, I present such conversions leading to exciting objects in mathematics.**

***Keywords*: Knight Tour Semi-Magic Square, Hamiltonian Path, Amino Acid bases, Latin Squares, Permutation Groups.**

1. Introduction

The problem of finding a Knight Tour on a conventional 8 × 8 chessboard so that the knight starts moving in its legal L – shaped form and cover the whole chessboard without leaving any square and should not land on the same square more than once is called "Knight Tour Problem." This classic puzzle was solved by many, including the great Swiss mathematician Leonhard Euler. Let us consider the Knight's Tour solution by William Beverly in 1848.

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

Figure 1

We notice that beginning at 1, moving through legal L-shaped steps for the Knight piece, we can move across the whole 64 squares in the chess board so that each square is visited only once. However, unfortunately, there is no way in Figure 1 to return to 1 from 64. Hence this tour is called as Open Knight Tour. A closed Knight Tour is a tour in which we can return to 1 from 64 after covering all the squares exactly once. These kinds of traversing problems are called "Hamiltonian Paths" in Graph Theory. The path of the Knight that traversedthe whole chessboard in Figure 1 is one such path. We also notice that the sum of numbers in any row or column adds up to 260, whereas the diagonals do not. In this sense, the Knight's Tour shown in Figure 1 only represent a semi-magic square but not a magic square.
Considering the Knight's Tour shown in Figure 1, I shall explore some mathematical properties in this paper to obtain new structures.

2. New from Old

In Figure 1, being an 8 × 8 square, we can subdivide it into four distinct 4 × 4 sub-squares, as shown in Figure 2.

1	48	31	50
30	51	46	3
47	2	49	32
52	29	4	45

33	16	63	18
62	19	14	35
15	34	17	64
20	61	36	13

5	44	25	56
28	53	8	41
43	6	55	26
54	27	42	7

9	40	21	60
24	57	12	37
39	10	59	22
58	23	38	11

Figure 2: Four sub squares of 8 × 8 squares in Figure 1

Each of these four distinct squares is a semi-magic square in that the sum of any row and column adds up to 130. Note that this number 130 is exactly half of the Magic sum 260 of the parent square in Figure 1.

3. Forming Latin Square Patterns

We now try to replace the 16 numbers in each square with DNA amino acid bases **A**, **T**, **G**, and **C** using a scheme illustrated below: In each of the 4 × 4 small sub-squares, if we now replace the amino acid base **A** for the numbers 1,2,3,4 in the firstsquare; 13, 14, 15,16 in the second square; 5,6,7,8 in the third square; 9,10,11,12 in the fourth square followed by replacing **T** for the numbers 29,30,31,32 in the first square; 17,18,19,20 in the second square; 25,26,27,28 in the third square; 21,22,23,24 in the fourth square; replacing **G** for the numbers 45,46,47,48 in the first square; 33,34,35,36 in the second square; 41,42,43,44 in the third square; 37,38,39,40 in the fourth square; replacing **C**for the numbers 49,50,51,52 in the first square; 61,62,63,64 in the second square; 53,54,55,56 in the third square; 57,58,59,60 in the fourth square respectively.
If we do these replacements for respective numbers by Amino Acid bases, we get four 4 × 4 sub-squares, as shown in Figure 3.

A	G	T	C
T	C	G	A
G	A	C	T
C	T	A	G

G	A	C	T
C	T	A	G
A	G	T	C
T	C	G	A

A	G	T	C
T	C	A	G
G	A	C	T
C	T	G	A

A	G	T	C
T	C	A	G
G	A	C	T
C	T	G	A

Figure 3

If we notice all four 4 × 4 squares in Figure 3, it is clear that they form Latin Squares. Each square contains precisely one letter among the four letters (Amino Acid Bases) **A**, **G**, **T**, and **C** exactly once in any row or column. Such arrangements are called "Latin Squares" in mathematics.

4. Forming Group Structures

We now compare the first two and the following two square entries of the four squares shown in Figure 3. Doing this, we see that the third and fourth squares are precisely the same having similar entries for all 16 cells of thesquares.
Similarly, if we look at the first two squares, the first row of the first square became the third row of the second square, the second row of the first square became the fourth row of the second square, the third row of thefirst square had become the first row of the second square, the fourth row of the first square had become the second row of the fourth square. Viewing this, we can form the following two-element set corresponding to these row arrangements described above.
The rows transform for the first two squares as $1 \to 3, 2 \to 4, 3 \to 1, 4 \to 2$.

For the third and fourth squares, the rows transform as $1 \to 1, 2 \to 2, 3 \to 3, 4 \to 4$.If we now make this row transforms as each element of the set ***E***, then we have

$$E = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \right\} = \{e, a\} \qquad (1)$$

Treating the row transforms as elements *e*, *a* of the set *E* and considering them as permutations of the numbers 1,2,3,4 under the composition of functions binary operation, we get the following Cayley's Table for the elements of *E*.

$\circ$	e	a
e	e	a
a	a	e

Figure 4

Looking at the table in Figure 4, we find that this is precisely the composition table for the permutation group S_2, a symmetric group with 2 elements. As S_2 is Abelian from the Hamiltonian Path given in Figure 1, we arrived at a Permutation group S_2 which is Abelian. We note that among Permutation groups, only S_1 and S_2 are Abelian Groups.
Hence our original knight tour chessboard problem has produced four Latin Square structures and a Permutation group with two

symbols, Abelian. Moreover, these conversions from the original chessboard move numbers to DNA bases and, in turn to Permutation groups, providing a new scope for exploring and understanding the process behind Hamiltonian Paths for higher-order square grids.

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