Edge Failure Sensitivity of Wiener and Harary Indices in Classical Network Graphs

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Abstract: Single-edge failure changes all-pairs distances in a highly structured way, but the resulting change in a distance-sum index and a reciprocal-distance index is rarely available in exact form. For a connected simple graph G and a non-bridge edge e, we study the Wiener sensitivity Δ_W (G,e)=W(G-e)-W(G) and the Harary sensitivity Δ_H(G,e)=H(G)-H(G-e). These quantities are not introduced as new network centralities: reciprocal-distance efficiency loss and normalized Wiener-based edge importance have clear precedents. The contribution is an exact joint theorem package. We characterize affected vertex pairs, obtain a replacement-path bound, and prove that whenever diam(G)≤2 and the failed edge has a two-step replacement, $\Delta_H=(\Delta_W+2)/6$. Closed forms are then derived for cycles, wheels, fans, and friendship graphs. Wheel spokes become more sensitive than rim edges from order 7 onward; friendship hub-incident edges dominate peripheral edges for at least two triangles; and fan spokes display symmetric position dependence that is not determined by endpoint degrees alone. The formulas were checked on every family edge through order 30, on all 995 connected unlabeled graphs of orders 2-7, and on additional fixed-seed random connected graphs, with no counterexamples to the stated theorems.

Keywords: Wiener index; Harary index; edge deletion; network vulnerability; wheel graph; fan graph; friendship graph.

I. INTRODUCTION

Distance-based graph indices summarize global structure using shortest paths. The Wiener index, originating in chemical graph theory, sums pairwise distances [1], while the Harary index sums their reciprocals [2]. They respond in opposite directions to a connectivity-preserving edge deletion: shortest-path distances can only increase, so the Wiener index increases and the Harary index decreases. This elementary monotonicity makes the two indices natural complementary lenses for a single-link failure.

The vulnerability interpretation itself is not new. Global network efficiency is a normalized sum of reciprocal distances [3], and removal-based information centrality evaluates the loss of such efficiency [4], [5]. Wiener-based edge importance has also been defined through the change of the Wiener index under deletion [6]. Applied network-resilience work has simultaneously examined average-path-length increase and global-efficiency loss after a component failure [7]. Consequently, the present paper does not claim to invent either the edge-failure idea or a new centrality measure.

The mathematical gap addressed here is narrower and more defensible: exact joint formulas and structural relations between the Wiener increase and the Harary decrease for explicit graph families. Recent work connects Wiener-type indices with vulnerability parameters [8]-[10], while exact distance-based indices are available for friendship and generalized friendship graphs [11], [12]. However, the searches performed for this project did not locate an existing theorem package that derives both single-edge sensitivities, classifies edge orbits or positions, and explains their coupling for cycles, wheels, fans and friendship graphs. The novelty claim is therefore limited to that exact theorem package rather than to the underlying vulnerability concept.

Our contributions are fourfold. First, we give an affected-pair decomposition and replacement-path bounds valid for every connected graph under non-bridge deletion. Second, we prove an affine coupling theorem, $\Delta_H=(\Delta_W+2)/6$, for diameter-at-most-two graphs when the deleted edge has replacement distance two. Third, we derive closed forms for the four classical families and obtain edge-class and position comparisons. Fourth, all formulas and general statements are subjected to exhaustive small-order falsification and additional fixed-seed stress testing. The computational layer is used only as a counterexample search and reproducibility aid; every reported theorem is proved analytically.

II. PRELIMINARIES AND SCOPE OF THE SENSITIVITY FRAMEWORK

All graphs in this paper are finite, simple, undirected and connected. For x,y∈V(G), d_G(x,y) is their shortest-path distance. The Wiener and Harary indices are

$$W(G) = \sum_{\{x,y\}\subseteq V(G)} d_G(x,y), \qquad H(G) = \sum_{\{x,y\}\subseteq V(G)} \frac{1}{d_G(x,y)}.$$

Let e=uv be a non-bridge edge, so G-e remains connected. We define the two signed failure responses

$$\Delta_W(G,e) = W(G-e) - W(G), \qquad \Delta_H(G,e) = H(G) - H(G-e).$$

Both are nonnegative by distance monotonicity; later we prove strict positivity. If n=|V(G)| and E_{glob} denotes the standard unweighted global efficiency, then E_{glob}(G)=2H(G)/[n(n-1)]. Hence Δ_H is exactly n(n-1)/2 times the absolute loss E_{glob}(G)-E_{glob}(G-e). This identity is why Δ_H should be viewed as an exact graph-theoretic form of an established efficiency-loss concept rather than as a new centrality [3]-[5]. Similarly, the normalized Wiener edge-importance quantity used by Tittmann and Kischnick [6] is a monotone normalization of the same increase measured here by Δ_W.

We use the following family conventions. C_n is the cycle of order n. W_n=K_1∨C_{n-1} is the wheel of order n (n≥4), with hub c. F_n=K_1∨P_{n-1} is the fan of order n (n≥3), whose path vertices are v_1,...,v_{n-1} in order and whose hub is c. Fr_k is the friendship graph formed from k triangles sharing exactly one common hub; it has 2k+1 vertices. The explicit convention is important because wheel and fan subscripts vary across the literature.

III. GENERAL STRUCTURE UNDER A NON-BRIDGE EDGE FAILURE

Lemma 1 (strict monotonicity). *If e is a non-bridge edge of a connected simple graph G, then Δ_W(G,e)>0 and Δ_H(G,e)>0.*

Proof. Deleting an edge cannot create a shorter path, so d_{G-e}(x,y)≥d_G(x,y) for every vertex pair. In particular, if e=uv then d_G(u,v)=1 whereas d_{G-e}(u,v)≥2. Therefore at least one distance increases strictly. Summing distances gives W(G-e)>W(G). Since t↦1/t is strictly decreasing for positive t, the reciprocal-distance contribution of every changed pair decreases and that of {u,v} decreases strictly, yielding H(G)>H(G-e). □

Lemma 2 (edge-orbit invariance). *If an automorphism of G maps edge e to edge f, then Δ_W(G,e)=Δ_W(G,f) and Δ_H(G,e)=Δ_H(G,f).*

Proof. An automorphism mapping e to f induces an isomorphism G-e≅G-f. Wiener and Harary indices are graph invariants, so both deleted graphs have the same pair of index values. Subtracting the common base values W(G) and H(G) proves the claim. □

For a non-bridge edge e, define the affected-pair set

$$A_e = \{\{x,y\} \subseteq V(G) : d_{G-e}(x,y) > d_G(x,y)\}.$$

Lemma 3 (geodesic characterization). *A pair {x,y} belongs to A_e if and only if every x-y geodesic in G uses e.*

Proof. If some shortest x-y path avoids e, that path survives in G-e and therefore d_{G-e}(x,y)≤d_G(x,y); monotonicity forces equality. Conversely, if every geodesic uses e, deletion destroys every path of length d_G(x,y), so the new distance is strictly larger. □

Theorem 1 (exact decomposition and replacement bounds). *Let e=uv be a non-bridge edge, let ℓ_e=d_{G-e}(u,v), and put D′=diam(G-e). For {x,y}⊆A_e write d=d_G(x,y), d′=d_{G-e}(x,y), and δ=d′-d. Then*

$$\Delta_W(G,e) = \sum_{\{x,y\}\in A_e} \delta_{xy}, \qquad \Delta_H(G,e) = \sum_{\{x,y\}\in A_e} \frac{\delta_{xy}}{d_G(x,y)\, d_{G-e}(x,y)}.$$

$$|A_e| \le \Delta_W(G,e) \le (\ell_e - 1)|A_e|.$$

$$\frac{\Delta_W(G,e)}{D'(D'-1)} \le \Delta_H(G,e) \le \frac{\Delta_W(G,e)}{2}.$$

Proof. Only affected pairs contribute to either difference, and 1/d-1/d′=(d′-d)/(dd′), giving the exact sums. For an affected pair, choose a geodesic in G; by Lemma 3 it uses e. Replace that occurrence of e by a shortest u-v path in G-e of length ℓ_e. This constructs an x-y walk of length d+ℓ_e-1, so 1≤δ≤ℓ_e-1. Summation yields the Wiener bounds. For every affected pair, d′≤D′ and d≤D′-1, hence dd′≤D′(D′-1), while d≥1 and d′≥2 give dd′≥2. Applying these product bounds termwise to δ/(dd′) proves the final inequalities. □

The decomposition is useful because it separates the combinatorial support of the failure from the magnitude of its rerouting cost. The set A_e depends on geodesic uniqueness relative to e: a vertex pair is unaffected whenever at least one original geodesic survives. By contrast, ℓ_e measures how expensive the local replacement of the failed edge can be. Thus two edges with the same number of affected pairs may still have different Wiener responses when their alternate u-v paths have different lengths.

Corollary 1.1 (two-step replacement). *If $d_{G-e}(u,v)=2$, then every affected pair increases by exactly one and therefore $\Delta_W(G,e)=|A_e|$.*

Proof. Theorem 1 gives 1≤δ≤ℓ_e-1=1 for every affected pair, hence δ=1 termwise. Summing the exact Wiener decomposition yields $\Delta_W=|A_e|$. □

Theorem 2 (diameter-two affine coupling). *Suppose diam(G)≤2 and e=uv is a non-bridge edge with $d_{G-e}(u,v)=2$. Let $q=|A_e|-1$. Then the endpoint pair changes 1→2, every other affected pair changes 2→3, and*

$$\Delta_W(G,e) = 1+q, \qquad \Delta_H(G,e) = \frac{1}{2}+\frac{q}{6}.$$

$$\Delta_H(G,e) = \frac{\Delta_W(G,e)+2}{6}.$$

Proof. The pair {u,v} contributes 1 to Δ_W and 1/2 to Δ_H. Consider any other affected pair {x,y}. It cannot have d_G(x,y)=1: if xy is an edge different from e, that edge remains and the pair stays at distance one. Because diam(G)≤2, it follows that d_G(x,y)=2. The replacement-path bound with ℓ_e=2 gives δ≤1, while affectedness gives δ≥1; thus d_{G-e}(x,y)=3. Each of the q remaining affected pairs therefore contributes 1 and 1/2-1/3=1/6, respectively. The displayed formulas follow. □

The hypotheses in Theorem 2 are substantive. Diameter two fixes every nonadjacent original pair at distance two, while the two-step replacement fixes the largest possible increase of an affected pair at one. Removing either hypothesis can destroy the affine relation: in a cycle, for example, the deleted edge has a replacement path whose length grows with n, and affected pairs can experience several different increments. The coupling theorem should therefore be read as a structural signature of short redundant routing rather than as a universal relation between Wiener and Harary indices.

Theorem 2 is the structural mechanism behind all wheel, fan and friendship formulas below: these graphs have diameter at most two and every considered edge lies in a triangle, so its endpoints retain a two-step replacement after deletion.

IV. CYCLES

Every edge of C_n lies in the same edge orbit. Deleting any edge turns C_n into P_n, so the sensitivities follow from standard distance counts. Let h_r denote the r-th harmonic number.

$$h_r = \sum_{j=1}^{r} \frac{1}{j}.$$

The cycle is also the natural contrast to the diameter-two families considered later. In C_n the alternate route between the endpoints of a failed edge has length n-1, so the replacement parameter is ℓ_e=n-1 rather than two. A single deletion removes the cyclic shortcut for many pairs, and the distance increments are not uniform. This is precisely the regime in which the affected-pair decomposition is more informative than simply counting changed pairs.

Theorem 3 (cycle sensitivities). *For any edge e of C_n, n≥3, the Wiener sensitivity is*

$$\Delta_W(C_n,e) = \frac{n(n^2-1)}{24}, \qquad n \text{ odd}.$$

$$\Delta_W(C_n, e) = \frac{n(n^2-4)}{24}, \qquad n \text{ even.}$$

For n=2m+1,

$$\Delta_H(C_{2m+1}, e) = 2m - (2m+1)\sum_{j=m+1}^{2m} \frac{1}{j}.$$

For n=2m,

$$\Delta_H(C_{2m}, e) = 2m\left(1 - \sum_{j=m}^{2m-1} \frac{1}{j}\right).$$

For completeness, the path identities can be obtained directly by grouping unordered vertex pairs according to their separation r. In P_n there are n-r pairs at distance r, hence

$$W(P_n) = \sum_{r=1}^{n-1} r\,(n-r) = \frac{n^3-n}{6}.$$

$$H(P_n) = \sum_{r=1}^{n-1} \frac{n-r}{r} = nh_{n-1} - (n-1).$$

For an odd cycle C_{2m+1}, each vertex sees exactly two vertices at every distance 1,...,m. Dividing the resulting ordered count by two gives

$$W(C_{2m+1}) = \frac{(2m+1)m(m+1)}{2} = \frac{n(n^2-1)}{8}, \qquad H(C_{2m+1}) = (2m+1)h_m.$$

For an even cycle C_{2m}, there are two vertices at each distance 1,...,m-1 and one antipodal vertex at distance m, yielding

$$W(C_{2m}) = \frac{n^3}{8}, \qquad H(C_{2m}) = 2mh_{m-1} + 1.$$

Proof. The path formulas are W(P_n)=(n^3-n)/6 and H(P_n)=n h_{n-1}-(n-1). For the cycle, W(C_n)=n(n^2-1)/8 for odd n and n^3/8 for even n. Also H(C_{2m+1})=(2m+1)h_m, while H(C_{2m})=2m h_{m-1}+1 because each vertex has two vertices at every distance 1,...,m-1 and one antipodal vertex at distance m. Subtracting W(C_n) from W(P_n) and H(P_n) from H(C_n), followed by the identities h_{2m}-h_m=$\sum_{j=m+1}^{2m}\frac{1}{j}$ and h_{2m-1}-h_{m-1}=$\sum_{j=m}^{2m-1}\frac{1}{j}$, gives the formulas. □

The cycle response illustrates that the two sensitivities need not grow on the same scale. As n tends to infinity,

$$\Delta_W(C_n, e) = \frac{n^3}{24} + O(n), \qquad \Delta_H(C_n, e) = n(1-\ln 2) + O(1).$$

Thus the absolute distance penalty is cubic while the reciprocal-distance loss is only linear.

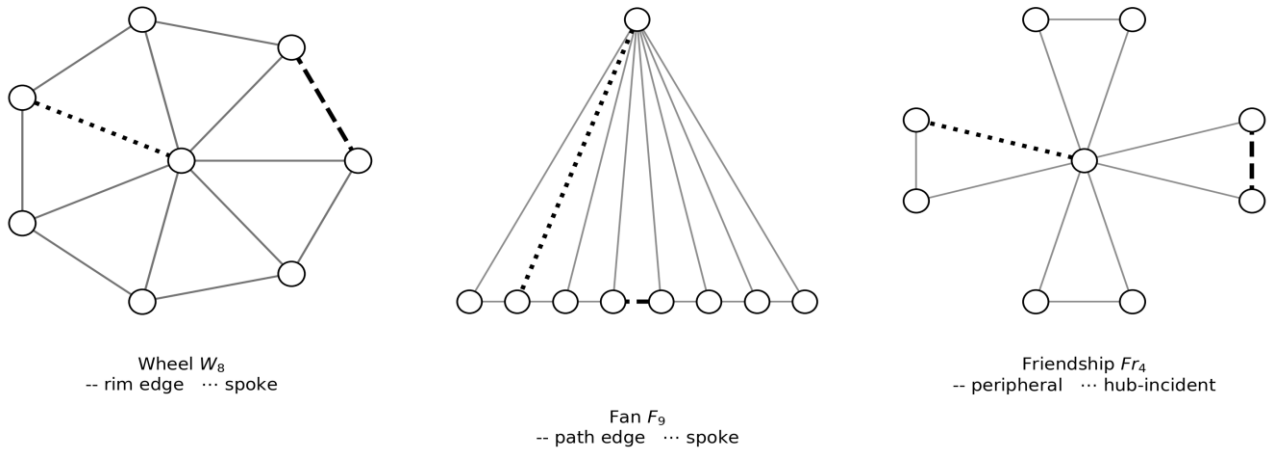


Fig. 1. Representative edge classes used in the wheel, fan and friendship calculations. Dashed and dotted edges indicate the two compared failure classes.

V. WHEELS, FANS AND FRIENDSHIP GRAPHS

A. Wheels

Theorem 4 (wheel edge classes). *Let W_n=K_1 ⨁ C_{n-1}, n≥4. For a rim edge r and a spoke s,*

$$\Delta_W(W_n, r) = 1, \qquad \Delta_H(W_n, r) = \frac{1}{2}.$$

$$\Delta_W(W_n, s) = 1 + \max\{0, n-6\}.$$

$$\Delta_H(W_n, s) = \frac{1}{2} + \frac{\max\{0, n-6\}}{6}.$$

Proof. A deleted rim edge has a two-step replacement through the hub. No other pair loses a geodesic, so q=0 in Theorem 2. For a spoke cv, the pair {c,v} changes 1→2. A rim vertex w that was at rim distance at least three from v had d_W(v,w)=2 through c; after cv is deleted, its shortest distance becomes three. Vertices at rim distance one or two retain paths of lengths one or two. On C_{n-1}, the number of rim vertices at cyclic distance at least three from v is max{0,n-6}. Theorem 2 now gives the formulas. □

Consequently, rim and spoke failures tie for 4≤n≤6; for n≥7 every spoke is strictly more sensitive than every rim edge under both indices. At n=4 the wheel is K_4, so the two descriptive classes are in fact one automorphism orbit, consistent with the equal values.

The threshold n=7 has a transparent geometric origin. Before that order, every rim vertex is at cyclic distance at most two from the endpoint v of a deleted spoke, so each such vertex retains an equally short route to v. Once the rim contains at least six vertices, positions at cyclic distance three appear. Those are exactly the vertices whose two-step hub geodesic depended uniquely on the deleted spoke, and each additional such vertex adds one unit to Δ_W and one-sixth to Δ_H.

B. Fans

Theorem 5 (fan path edges and positional spokes). *Let F_n=K_1 ⨁ P_{n-1}. Every path edge p satisfies (Δ_W,Δ_H)=(1,1/2). For the spoke s_i=cv_i, 1≤i≤n-1, put*

$$q_i = \max\{0, i-3\} + \max\{0, n-i-3\}.$$

Then

$$\Delta_W(F_n, s_i) = 1 + q_i, \qquad \Delta_H(F_n, s_i) = \frac{1}{2} + \frac{q_i}{6}.$$

Proof. Deleting a path edge v_i v_{i+1} leaves the two-step route v_i-c-v_{i+1}; all other pair distances are unchanged, so Theorem 2 gives (1,1/2). For a spoke cv_i, the endpoint pair changes 1→2. A path vertex v_j with |i-j|=1 remains adjacent to v_i, and one with |i-j|=2 remains at distance two along the path. If |i-j|≥3, the original distance two path v_i-c-v_j is destroyed and the best surviving route has length three, either by moving one step along the path to a vertex still adjacent to c or by another equivalent route. The number of indices j with |i-j|≥3 is exactly max{0,i-3}+max{0,n-i-3}=q_i. Theorem 2 completes the calculation. □

The spoke sensitivity is symmetric: q_i=q_{n-i}. For n≥4, the maximum occurs at the two endpoint spokes and equals q=n-4, so their sensitivities are Δ_W=n-3 and Δ_H=(n-1)/6. The minimum is q=max{0,n-6}; for n≥6 it is attained by positions i=3,...,n-3. Thus the hub does not make all spokes equivalent: distance from the path boundary matters.

This positional profile has a plateau-and-boundary structure. Moving a spoke endpoint inward from v_1 to v_2 and then v_3 removes one far-side dependent pair at each step. Once the endpoint is at least two path edges from both boundaries, the counts contributed from the left and right compensate, producing the constant central plateau q_i=n-6 whenever that expression is positive. The profile is therefore symmetric without being edge-transitive.

Corollary 5.1 (endpoint degrees do not determine sensitivity). For every n≥7, the spokes cv_2 and cv_3 have the same endpoint-degree pair (n-1,3), but different sensitivities:

$$\Delta_W(F_n, cv_2) - \Delta_W(F_n, cv_3) = 1, \qquad \Delta_H(F_n, cv_2) - \Delta_H(F_n, cv_3) = \frac{1}{6}.$$

Proof. Both v_2 and v_3 are internal path vertices of degree three, while c has degree n-1. The formula of Theorem 5 gives q_2=n-5 and q_3=n-6 for n≥7. Subtraction gives the stated differences. □

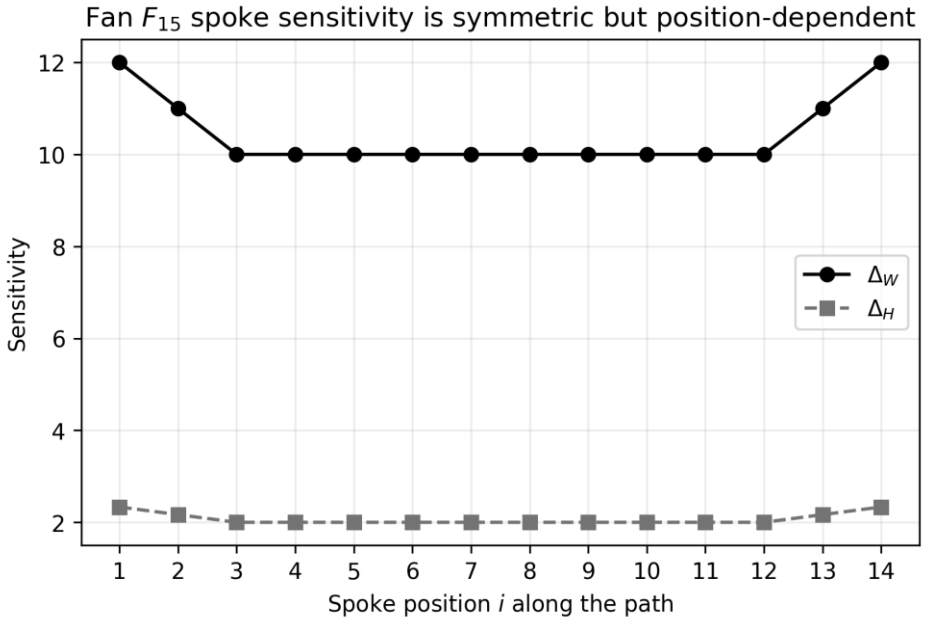


Fig. 2. Position dependence of spoke sensitivities in F_{15}. The symmetry i↔15-i and the central plateau follow directly from q_i.

C. Friendship graphs

Theorem 6 (friendship edge classes). *Let* Fr_k *consist of k triangles with common hub c. For a peripheral edge p within one triangle and a hub-incident edge s,*

$$\Delta_W(Fr_k, p) = 1, \qquad \Delta_H(Fr_k, p) = \frac{1}{2}.$$

$$\Delta_W(Fr_k, s) = 2k - 1, \qquad \Delta_H(Fr_k, s) = \frac{2k+1}{6}.$$

Proof. A peripheral edge has a two-step replacement through c, and no other pair changes. For a hub-incident edge ca in one triangle, {c,a} changes 1→2. The companion vertex b of the same triangle remains adjacent to a. For each of the other k-1 triangles, both of its peripheral vertices were at distance two from a through c and become distance three after ca is removed. Therefore q=2(k-1), and Theorem 2 yields Δ_W =1+2(k-1)=2k-1 and Δ_H =1/2+2(k-1)/6=(2k+1)/6. □

For k=1, $\mathrm{Fr}_1 = C_3$ and all edges tie. For every k≥2, hub-incident edges are strictly more sensitive than peripheral edges under both measures. The base Wiener index $W(\mathrm{Fr}_k)=4k^2-k$ is consistent with the friendship-graph distance count reported in [11], while recent generalized-friendship work also confirms the relevance of exact distance-index formulas for this family [12].

The linear growth of the hub-edge response has a different mechanism from the fan boundary effect. Each new triangle contributes two new peripheral vertices, and both rely on the hub edge ca for their unique length-two route to a. Thus adding one triangle increases Δ_W for ca by exactly two and Δ_H by exactly one-third, while the sensitivity of a peripheral edge remains constant. This clean separation makes Fr_k a useful centralized benchmark for the affected-pair viewpoint.

VI. CROSS-FAMILY COMPARISON AND STRUCTURAL INTERPRETATION

Table 1 collects the exact responses in a common notation. The table makes two distinctions visible. Cycles have only one edge orbit and a long replacement path, producing much larger growth of Δ_W. The other three families have diameter at most two and triangular edge replacement, which forces the affine coupling of Theorem 2; their variation is therefore controlled entirely by the count q of additional vertex pairs that change from distance two to distance three.

Family / edge class	Parameter	Δ_W	Δ_H
C_n / any edge	n odd	$\frac{n(n^2-1)}{24}$	$nh_{(n-1)/2} - nh_{n-1} + n - 1$
C_n / any edge	n even	$\frac{n(n^2-4)}{24}$	$n\left(1 + h_{n/2-1} - h_{n-1}\right)$
W_n / rim	n≥4	1	$\frac{1}{2}$
W_n / spoke	n≥4	$1 + \max\{0, n-6\}$	$\frac{1}{2} + \frac{\max\{0,n-6\}}{6}$
F_n / path edge	n≥3	1	$\frac{1}{2}$
F_n / spoke cv_i	1≤i≤n-1	$1 + q_i$	$\frac{1}{2} + \frac{q_i}{6}$
Fr_k / peripheral	k≥1	1	$\frac{1}{2}$
Fr_k / hub-incident	k≥1	$2k-1$	$\frac{2k+1}{6}$

Table 1. Closed-form single-edge sensitivities. Here h_r is the r-th harmonic number and q_i=max(0,i-3)+max(0,n-i-3).

The quantity $|A_e|$ has a direct structural meaning: it counts vertex pairs whose complete set of geodesics is destroyed by the edge failure. The replacement distance ℓ_e controls the maximum amount by which each affected pair can stretch. Theorem 1 therefore separates two causes of vulnerability that a single edge statistic can otherwise mix: how many shortest-path relations depend exclusively on e, and how costly the best replacement path is.

In the diameter-two triangular setting, replacement cost is fixed at one extra hop. This removes the second source of variation and makes q the sole parameter. The affine relation between Δ_W and Δ_H is not a numerical coincidence of wheels, fans and friendship graphs; it is inherited from the pair transition pattern 1→2 for the failed endpoints and 2→3 for every other affected pair.

The fan result also cautions against purely local rankings. The equal endpoint degrees of cv_2 and cv_3 for n≥7 do not imply equal failure impact. What differs is the set of distant path vertices whose only length-two geodesic uses the failed spoke. In this sense, the sensitivities capture shortest-path dependency that local degree information alone misses.

VII. COMPUTATIONAL FALSIFICATION AND REPRODUCIBILITY

A Python/NetworkX verification program accompanies the manuscript. Harary values are accumulated with exact rational arithmetic rather than floating-point tolerances. The script regenerates every family, deletes each admissible edge, recomputes all-pairs shortest-path distances, and compares the observed responses with the closed forms. It also tests the general inequalities and the diameter-two affine identity.

The verification routine intentionally recomputes W and H from shortest-path dictionaries rather than deriving them from the theorem formulas being tested. For each base graph, it records all pairwise distances, rejects bridge deletions, constructs G-e, recomputes the distance dictionary, and only then forms the two index differences. Family labels and spoke positions are generated independently from those numerical index computations. This separation reduces the risk of validating a formula with code that silently reuses the same algebra.

Three falsification layers were used. The first targets every structurally classified family edge through order 30. The second enumerates all connected unlabeled Graph Atlas graphs through order seven and tests every non-bridge

edge, giving a topology-diverse exhaustive small-order check. The third uses 100 connected fixed-seed random graphs of orders 8–15 to exercise larger, less symmetric structures. Exact rational arithmetic is used for every Harary comparison.

Verification block	Objects tested	Admissible deletions/checks	Failures
Family formulas	C_n, W_n, F_n through order 30; Fr_k up to order 29	2,481 edge-formula checks	0
Graph Atlas exhaustive	All connected unlabeled graphs of orders 2-7	9,909 non-bridge edge deletions	0
Affine-coupling subset	Atlas cases with diam(G)≤2 and ℓ_e=2	4,890 edge deletions	0
Random stress	100 connected graphs, orders 8-15, seed 20260928	2,756 non-bridge edge deletions	0
Random affine subset	Same stress set, applicable cases	1,289 edge deletions	0

Table 2. Counterexample-search coverage used before manuscript finalization.

The exhaustive atlas layer contains 995 connected unlabeled graphs after excluding the one-vertex graph. Across 9,909 non-bridge edge deletions, no violation was found for strict positivity, the decomposition, the replacement-path bound, or the Wiener-Harary bounds. The affine identity was separately exercised on 4,890 applicable atlas edge deletions with no failure. The random layer adds 2,756 admissible deletions, including 1,289 affine-coupling cases, also with zero failures. These tests are not presented as proofs; their role is to expose overlooked small-order exceptions, notation errors, parity errors and faulty equality cases before submission.

VIII. DISCUSSION

The literature audit materially changes how these results should be interpreted. Reciprocal-distance efficiency loss after edge removal already appears in network centrality and resilience studies [3]-[7], and Wiener-type indices have a substantial vulnerability literature [8]-[10]. The defensible contribution is therefore not the act of deleting an edge and observing W or H. It is the exact simultaneous analysis of the two responses, the affected-pair formulation, the replacement-path bounds, the affine coupling theorem, and the resulting closed-form classification of the chosen graph families.

The four families illustrate complementary regimes. A cycle has no local redundancy beyond the long alternate arc; one failed edge converts C_n into P_n and affects many distances by more than one. Wheels, fans and friendship graphs possess dense two-step redundancy through a hub or triangle structure. Their affected distances move only by one hop, making sensitivity primarily a question of how many pairs depended on the failed edge. This distinction suggests that replacement distance and geodesic dependency are useful explanatory variables even when one does not use the Wiener or Harary indices as operational network metrics.

A second structural conclusion concerns agreement between the two rankings. In the diameter-two families covered by Theorem 2, Δ_H is an increasing affine function of Δ_W, so the two indices necessarily rank admissible edge failures identically. The agreement is therefore proved, not empirical. Outside this regime there is no such conclusion: the Harary response discounts changes involving already-distant pairs through the factor $1/(dd')$, whereas the Wiener response weights every added hop equally. Cycles already display the resulting difference in growth scale.

There are also clear limitations. The study treats unweighted simple graphs and a single non-bridge failure. Bridge deletion disconnects the graph and requires a convention for infinite distances or componentwise indices. Weighted graphs can break the integer transition patterns used in Theorem 2. Multiple simultaneous failures can interact non-additively because the first deletion changes the geodesic structure seen by the second. These directions are deliberately left outside the present rapid-scope paper rather than being claimed without proof.

IX. CONCLUSION

This paper gives a compact exact theory for how Wiener and Harary indices respond to a single connectivity-preserving edge failure. The affected-pair decomposition identifies precisely which shortest-path relations matter; the replacement-path bounds quantify how far they can stretch; and the diameter-two theorem converts an entire class of failures into the affine law $\Delta_H=(\Delta_W+2)/6$. These structural statements yield closed forms for cycles, wheels, fans and friendship graphs and reveal threshold, edge-class and position effects that are hidden by base index values alone. Exhaustive small-order and broader fixed-seed computational tests found no counterexample to any stated result. The scope is intentionally narrow: the work does not propose a new centrality measure, but supplies exact graph-theoretic formulas and proofs for established distance and reciprocal-distance failure responses.

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Data and code availability: No external dataset is required. The reproducibility package contains the verification source code, exact machine-readable outputs, figures and failure log used for the computational checks reported in Table 2.

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