

Fixed-Point Theory in Metric, Partially Ordered, and Orthogonal Metric-like Spaces with Applications.

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Abstract:

This paper aims to report detailed results on fixed points in classical, partial order, and some more generalized metric spaces such as and the resulting from-deduced geometric properties for or, orthogonally usual metric, without any additional assumptions. After introducing the basics, such as Banach's contraction principle, the investigation turns to non-expansive mappings and Geraghty contractions, with proofs of existence, uniqueness, and convergence. The text, which can serve as an introductory-to-advanced course or research-level material, connects the (modern) theory of function spaces with methods used in boundary value problems, DAEs, and optimization, including best approximation results. This synthesis highlights the ubiquity and power of fixed-point theory in many mathematical settings, while also underlining its usefulness for various theoretical and practical applications. This supports further advances in computational methods and nonlinear analysis in the research of more sophisticated types of spaces and complex practical problems.

Keyword: Fixed-point theorem, Geraghty-type contraction, Partial order metric spaces, Metric-like spaces, Orthogonal metric spaces, Contraction mappings.

1. INTRODUCTION

Fixed-point theory is a branch of mathematical analysis that studies the conditions under which a function generates fixed points (points that map to themselves). This theory has grown in scope ever since and has expanded to non-expansive mappings, weak contractions, and many other generalized settings to incorporate more sophisticated mathematical structures. The study of partial order metric spaces, where the orders yield naturally induced broad topologies in alignment with the metric, defines powerful techniques for the analysis of monotone operators and solutions to problems wherein natural element comparisons are involved, such as in differential equations and matrix inequalities. Generalized metric structures, such as metric-like, orthogonal metric-order, and conical geodesic bicombing spaces, are more general metrics from the classical point of view, as they support weakened versions of the classical metric properties that permit wider collections of fixed-point theorems to be formulated. These results have far-reaching implications for problems in functional analysis, optimization, and differential equations,

including the construction of solutions, especially existence and uniqueness, iterative schemes for equilibrium/optimization problems, and boundary or initial value problems balancing both abstract theory and real feasibility.

2. Preliminaries

2.1 Basic Definitions of Metric, Metric-like, and Partial Order Metric Spaces [Frechet, M. (1906) & Matthews, S. G. (1994)]:

A metric space (X, d) satisfies:

1. $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

A metric-like space (X, ρ) [Amini-Harandi] satisfies:

1. $\rho(x, y) = 0 \Rightarrow x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

A partial order metric space is $(X, d, \leq)$ where $\leq$ is a partial order on X .

2.2. Orthogonal Metric and Orthogonal Metric-like Spaces [Gordji, M. E., Rameani, M., De La Sen, M., & Cho, Y. J. (2017)]:

An orthogonal set $(X, \perp)$: $\exists x_0 \in X$ s.t. $\forall y \in X, x_0 \perp y$ or $y \perp x_0$.

An orthogonal metric space: $(X, \perp, d)$ where d is a metric and sequences preserve $\perp$ -relation.

An orthogonal metric-like space: $(X, \perp, \rho)$ with ρ metric-like

2.3. Geraghty Contraction [Geraghty, M.A. (1973)]:

Let (X, d) be a metric space and let $T: X \rightarrow X$ be a self-mapping. Then T is called a Geraghty contraction if there exists a function $\beta: [0, \infty) \rightarrow [0, 1)$

such that $\beta(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0$,

and $d(Tx, Ty) \leq \beta(d(x, y)) d(x, y)$ for all $x, y \in X$.

2.4 Monotone Mappings and Partial Order Compatibility: [Ran and Reurings (2004)]:

Let $(X, \leq)$ be a partially ordered set. A mapping $T: X \rightarrow X$ is said to be monotone (or order-preserving) if $x \leq y \Rightarrow T(x) \leq T(y)$ for all $x, y \in X$.

The partial order $\leq$ is said to be compatible with a metric d on X if the order intervals

$$[x, \infty) = \{z \in X: x \leq z\}$$

and

$$(-\infty, y] = \{z \in X: z \leq y\}$$

are closed subsets of (X, d) for every $x, y \in X$.

2.5. Picard Iteration (Non-expansive Mappings [Banach (1922)]):

For a self-mapping $T: X \rightarrow X$, the Picard iterative sequence $\{x_n\}$ is defined by

$$x_{n+1} = T(x_n), n \geq 0.$$

2.6. Weak convergence in Hilbert/uniformly convex spaces [Mann (1953); Krasnosel'skii (1955)]:

Let $T: X \rightarrow X$ be a self-mapping and let $\{\alpha_n\} \subset [0,1]$. The Krasnosel'skii–Mann iterative sequence is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), n \geq 0.$$

2.7. (Geraghty Contraction) [Geraghty, M. A. (1973)]:

Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is a Geraghty contraction if there exists, $[0, \infty) \rightarrow [0,1]$, nondecreasing, with $\beta(t) < 1$ for $t > 0$, such that

$$d(Tx, Ty) \leq \beta(d(x, y)) \cdot d(x, y), \forall x, y \in X$$

Lemma 2.8. [Browder (1967); Goebel and Kirk (1990)]:

Let X be a Banach space and let $T: X \rightarrow X$ be a nonexpansive mapping. Suppose that a sequence $\{x_n\} \subset X$ satisfies $x_n \rightharpoonup x$ and $x_n - Tx_n \rightarrow 0$.

Then $Tx = x$.

That is, the operator $I - T$ is demi-closed at 0.

3. Main Results:

Theorem 3.1. Let (X, d) be a complete metric space and $T: X \rightarrow X$ satisfy

$$\exists k \in [0,1): d(Tx, Ty) \leq kd(x, y), \forall x, y \in X.$$

Then:

1. T has a unique fixed point $x^* \in X$, i.e., $T(x^*) = x^*$.
2. For any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ converges to x^* .
3. The convergence estimate is

$$d(x_n, x^*) \leq \frac{k^n}{1 - k} d(x_1, x_0)$$

Proof: Let $x_0 \in X$ be arbitrary and define $x_{n+1} = T(x_n)$ for $n \geq 0$. Then, by the contraction property,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}).$$

By induction,

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0), \forall n \geq 0.$$

For $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) \\ &= k^n \frac{1 - k^{m-n}}{1 - k} d(x_1, x_0) \end{aligned}$$

As $n \rightarrow \infty$, $d(x_m, x_n) \rightarrow 0$, so $\{x_n\}$ is Cauchy. Since X is complete, $x_n \rightarrow x^* \in X$.

Taking limits in $x_{n+1} = T(x_n)$ and using continuity of T ,

$$x^* = T(x^*),$$

so x^* is a fixed point.

For uniqueness, suppose y^* is another fixed point. Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*)$$

Since $0 \leq k < 1$, this implies $d(x^*, y^*) = 0 \Rightarrow x^* = y^*$.

The convergence estimate follows from

$$d(x_n, x^*) \leq \sum_{i=n}^{\infty} d(x_{i+1}, x_i) \leq \frac{k^n}{1 - k} d(x_1, x_0)$$

Theorem 3.2. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a Geraghty contraction. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the Picard iterative sequence $x_{n+1} = T(x_n)$, $n \geq 0$, converges to x^* .

Proof.

Let $x_0 \in X$ be arbitrary and define the sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$, $n \geq 0$.

Using the Geraghty contractive condition, we obtain

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \beta(d(x_n, x_{n-1})) d(x_n, x_{n-1}).$$

Since $0 \leq \beta(t) < 1$ for $t > 0$, the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and converges to some $r \geq 0$. By the defining property of β , it follows that $r = 0$. Hence, $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0$.

Using standard arguments, one shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$.

Passing to the limit in $x_{n+1} = T(x_n)$, we obtain $T(x^*) = x^*$.

Thus x^* is a fixed point of T .

For uniqueness, suppose that y^* is another fixed point of T . Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \beta(d(x^*, y^*)) d(x^*, y^*).$$

Since $\beta(d(x^*, y^*)) < 1$, it follows that $d(x^*, y^*) = 0$,

and therefore $x^* = y^*$.

Hence T possesses a unique fixed point and the Picard iteration converges to it.

4. Applications:

4.1 Existence and Uniqueness of Solutions to Boundary Value Problems

Many boundary value problems (BVPs) for ordinary and partial differential equations can be transformed into operator equations $Tx = x$ in suitable Banach or metric-like spaces. If the associated operator T is a contraction, monotone contraction, or fits within generalized frameworks (e.g., orthogonal metric spaces), the existence and uniqueness of solutions follow directly from fixed point theorems. For example, the second-order BVP

$$x''(t) = f(t, x(t)), x(0) = \alpha, x(1) = \beta$$

can be reformulated via Green's function into an integral equation $x = Tx$, where T is a contraction under appropriate conditions on f .

4.2 Applications in Differential Equations

Fixed point principles apply to initial value problems, delay differential equations, and fractional differential equations. If $F: X \rightarrow X$ represents the solution mapping in a function space, conditions such as Geraghty contraction or $(\hbar, \varphi)$ -contraction guarantee not only existence but also stability of solutions. In particular, Nonlinear functional differential equations of the form

$$x'(t) = g(t, x(t), x(\tau(t)))$$

can be handled in ordered metric spaces where monotonicity of g ensures applicability of monotone mapping theorems.

4.3 Optimization and Best Approximation Problems:

In optimization, the minimizer of a convex functional often corresponds to a fixed point of its resolvent or a proximal mapping. In Hilbert or Busemann spaces, non-expansive mappings represent projection operators P_C onto closed convex sets C , where fixed point results yield the existence and uniqueness of best approximations:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|$$

Similarly, iterative methods, such as Halpern or Krasnosel Mann iterations, can compute such approximations, providing practical algorithms for solving convex feasibility and variational inequality problems.

5. Conclusion

In this chapter, we studied several important fixed-point results in metric spaces and their generalizations. Beginning with the classical Banach Contraction Principle, we established the existence, uniqueness, and convergence of fixed points under various contractive conditions. We further examined non-expansive mappings, Geraghty contractions, and Matkowski-type contractions, highlighting their role as natural extensions of the Banach framework.

The results presented demonstrate the effectiveness of iterative methods, particularly Picard and Krasnosel'skii–Mann iterations, in approximating fixed points. These developments illustrate the flexibility of fixed-point theory in dealing with broader classes of mappings beyond classical contractions and provide a useful foundation for further investigations in nonlinear analysis, optimization, and applied mathematics.

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