

Generalized BRST transformation for relating Proca and Stueckelberg theory

Ghanshyam Pal¹

P.G. Department of Physics, Magadh University, Bodh-Gaya

Ajay Kumar Singh

Department of Physics, A.M. College,

Gaya, Magadh University, Bodh-Gaya

This work explores the relationship between the generating functionals associated with first-class and second-class constrained systems through the application of the finite field-dependent BRST (FFBRST) transformation. We rigorously demonstrate this correspondence by carrying out explicit computations within the framework of two distinct models. In particular, we derive the generating functional corresponding to the Proca theory by systematically transforming the generating functional of the Stueckelberg formulation for a massive spin-1 vector field via the FFBRST approach. This analysis underlines the utility of FFBRST transformations in establishing a formal equivalence between different quantization schemes.

I. INTRODUCTION

The Dirac procedure for analyzing constrained systems has been extensively employed within the Hamiltonian framework to facilitate the quantization of systems characterized by second-class constraints [1]. Despite its utility, the resulting Dirac brackets—integral to this approach—tend to exhibit field dependence and nonlocal behavior. These features introduce significant complications, notably operator ordering ambiguities, which obstruct the identification of canonically conjugate dynamical variables. Conversely, the quantization of systems with first-class constraints [2] has been successfully achieved in a gauge-invariant framework that preserves BRST symmetry [3–5]. Notably, systems governed by second-class constraints can be reformulated as equivalent first-class systems within an appropriately extended phase space [2–6]. This methodology, systematically developed by Batalin, Fradkin, and Tyutin [7, 8], has been widely adopted and applied to a broad spectrum of models [9–12].

In four-dimensional spacetime (1+3), the Proca model, which depicts a massive spin-1 vector field, features second-class constraints due to the explicit mass term breaking gauge symmetry [13–17]. To address this, the Stueckelberg mechanism introduces an auxiliary scalar field, effectively restoring gauge invariance and converting the system into a firstclass framework [18–20]. This gauge-invariant

¹ e-mail address: ghanshyampalindian12345@gmail.com

representation of massive spin-1 fields has broad relevance in modern gauge field theory and various contexts within string theory

[21–23].

This paper explores how theories featuring first-class constraints can be connected to those governed by second-class constraints via the FFBRST transformation, as proposed by Joglekar and Mandal [24]. Unlike the standard BRST transformation, FFBRST employs a finite, field-sensitive, anticommuting parameter, making it a broader framework. Despite being nilpotent and preserving the symmetry of the effective action, the FFBRST transformation modifies the path integral measure and hence alters the generating functional. Its effectiveness is well-illustrated through multiple applications across various field-theoretic contexts [24–32]. A parallel development, the finite field-dependent anti-BRST transformation, demonstrates similar capabilities [27, 29]. In this study, we illustrate how the generating functional of the Stueckelberg formulation for a massive spin-1 vector field is transformable into that of the Proca model using both FFBRST and FF-anti-BRST methods. A comparable link is also shown between the gauge-invariant and gauge-dependent versions of the self-dual chiral boson model. This approach offers a resolution to the difficulties caused by nonlocal, field-sensitive Dirac brackets in secondclass quantization, by enabling a connection to first-class theories where Green's functions are more tractable.

This paper is organized as follows. Section II introduces the concept behind the FFBRST transformation. In Section III, we provide a concise review of models with second-class and first-class constraints, particularly those describing a massive spin-1 field. Section IV demonstrates how the FFBRST method connects the Proca model with the Stueckelberg framework. Lastly, Section V presents our discussion and final remarks.

II. EXTENDED BRST TRANSFORMATIONS

This section delves into the framework of BRST transformations, beginning with standard properties that hold irrespective of whether the transformation parameter Λ is finite or infinitesimal, or whether it depends on fields or not—provided it is Grassmann-valued and constant in space-time. These conditions permit the expansion of BRST transformations by letting Λ become both finite and field-dependent, without compromising its key attributes.

To build such an extension, we start by allowing the infinitesimal BRST parameter to depend on fields. For ease in formulation, we introduce a continuous interpolation parameter κ within the range $0 \leq \kappa \leq 1$, and redefine each field as a κ -dependent entity, $\phi(x, \kappa)$. The setup ensures that $\phi(x, 0) = \phi(x)$ (original field) and $\phi(x, 1) = \phi'(x)$ (transformed field).

The infinitesimal form of the BRST transformation for this κ -dependent framework becomes

$$d\phi(x, \kappa) = \delta_b[\phi(x, \kappa)]\Theta'[\phi(x, \kappa)]d\kappa, \quad (2.1)$$

where $\Theta'[\phi(x, \kappa)]d\kappa$ denotes a field-dependent, infinitesimal shift.

The finite field-dependent BRST parameter, in turn, arises from integrating this infinitesimal form over the entire κ domain:

$$\phi'(x) = \phi(x) + \delta_b[\phi(x)]\Theta[\phi(x)], \quad (2.2)$$

with

$$\Theta[\phi(x)] = \int_0^1 d\kappa' \Theta'[\phi(x, \kappa')] \quad (2.3)$$

This type of field-dependent finite BRST (FFBRST) transformation maintains nilpotency and remains a symmetry of the effective action. However, due to the parameter's finite nature, the path integral measure does not retain its invariance under FFBRST. The associated Jacobian can still be determined for specific choices of the parameter $\Theta[\phi(x)]$. The transformation law for the measure is expressed as:

$$D\phi = J(\kappa) D\phi(\kappa) = J(\kappa + d\kappa) D\phi(\kappa + d\kappa). \quad (2.4)$$

Since the shift $\phi(\kappa) \rightarrow \phi(\kappa + d\kappa)$ is infinitesimal, the Jacobian ratio is given by:

$$\frac{J(\kappa)}{J(\kappa + d\kappa)} = \Sigma_\phi \pm \frac{\delta\phi(x, \kappa)}{\delta\phi(x, \kappa + d\kappa)}, \quad (2.5)$$

where the sum extends over all contributing fields, and the $\pm$ reflects whether ϕ is fermionic or bosonic.

This leads to a differential expression for the infinitesimal change in $J(\kappa)$:

$$\frac{1}{J} \frac{dJ}{d\kappa} = - \int d^4x \left[\pm \delta_b \phi(x, \kappa) \frac{\partial \Theta'[\phi(x, \kappa)]}{\partial \phi(x, \kappa)} \right], \quad (2.6)$$

Under certain conditions, the Jacobian $J(\kappa)$ may be replaced within the functional integral by an exponential factor involving a local action:

$$J(\kappa) \rightarrow \exp[iS_1[\phi(x, \kappa)]], \quad (2.7)$$

provided the following criterion holds [24]:

$$\int D\phi(x) \left[\frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1[\phi(x, \kappa)]}{d\kappa} \right] \exp[i(S_{eff} + S_1)] = 0. \quad (2.8)$$

Here, $S_1[\phi]$ is a local functional depending on the fields.

By choosing suitable forms of Θ , one can ensure that the transformed functional,

$$Z' \equiv \int D\phi e^{i(S_{eff} + S_1)}$$

represents another consistent effective theory connected through FFBRST.

III. EXAMPLES: THEORIES WITH CONSTRAINTS

We now describe key aspects of theories characterized by constraints. Specifically, we analyze models with second-class constraints such as the Proca theory (massive spin-1 vector field) and a non-invariant theory of a self-dual chiral boson. Their corresponding first-class versions are, respectively, the Stueckelberg formulation for massive vector bosons and the gauge-invariant version of the chiral boson model. Both classes of models are examined to showcase the implications of BRST symmetry in constrained systems.

1. Proca Framework

Let us consider the formulation describing a neutral, massive, spin-1 vector boson A_μ propagating in four-dimensional spacetime. The corresponding action is expressed as:

$$S_P = \int d^4x \mathcal{L}_P \quad (3.1)$$

with the associated Lagrangian density given by

$$\mathcal{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu. \quad (3.2)$$

The antisymmetric field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We adopt the Minkowski metric with the signature $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, where $\mu, \nu = 0, 1, 2, 3$ denote spacetime indices.

To determine the canonical structure, we evaluate the momentum conjugate to the field component A_μ :

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}. \quad (3.3)$$

From the above relation, one identifies the primary constraint of the system as:

$$\Omega_1 \equiv \Pi^0 = 0. \quad (3.4)$$

Now, the Hamiltonian density $\mathcal{H}$ can be constructed as follows:

$$\begin{aligned} \mathcal{H} &= \Pi_\mu \dot{A}^\mu - \mathcal{L} \\ &= \Pi_i \partial^i A^0 - \frac{1}{2} \Pi_i^2 + \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} M^2 A_\mu A^\mu. \end{aligned} \quad (3.5)$$

To examine the time stability of Π^0 , we compute its Poisson bracket with the full Hamiltonian $H = \int d^3x \mathcal{H}$:

$$\dot{\Pi}^0 = [\Pi^0, H] \quad (3.6)$$

Requiring consistency of the constraints over time yields the secondary relation:

$$\Omega_2 \equiv \partial_i \Pi^i + M^2 A^0 = 0. \quad (3.7)$$

Due to the appearance of A^0 in Ω_2 , the Poisson bracket between Ω_1 and Ω_2 does not vanish:

$$[\Omega_1, \Omega_2] \neq 0.$$

Thus, within this model, the constraints are categorized as second class.

Finally, the path integral representation for the Proca field theory is defined by the generating functional:

$$Z_P \equiv \int \mathcal{D}A_\mu e^{iS_P}. \quad (3.8)$$

2. Modified Viewpoint: Stueckelberg Model

To bypass the limitations of the Proca setup, the Stueckelberg framework employs an alternative Lagrangian form:

$$S_{ST} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left(A_\mu - \frac{1}{M} \eta_\mu B \right)^2 \right], \quad (3.9)$$

with the field B introduced as a real-valued scalar.

This structure respects the gauge transformation rules:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x), \quad (3.10)$$

$$B(x) \rightarrow B'(x) = B(x) + M\lambda(x), \quad (3.11)$$

where λ is a function acting as the gauge degree of freedom.

To proceed with quantization, one imposes a gauge-fixing term. Adopting the 't Hooft choice:

$$\mathcal{L}_{gf} = -\frac{1}{2\chi} (\partial^\mu A_\mu + \chi M B)^2,$$

where χ is a free gauge constant, one ensures that the propagators remain regular in the high-energy regime. Hence, renormalizability is not hindered.

We next examine BRST invariance within this scheme. To facilitate it, we introduce ghost (ω) and antighost (ω^*) fields. The resulting effective action becomes:

$$S_{ST} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left(A_\mu - \frac{1}{M} \eta_\mu B \right)^2 - \frac{1}{2\chi} (\eta^\mu A_\mu + \chi M B)^2 - \omega^* \left(\eta_\mu \partial^\mu + \chi M^2 \right) \omega \right]. \quad (3.12)$$

This action remains invariant under the BRST shift:

$$\begin{aligned}\delta_b A_\mu &= \partial_\mu \omega \Lambda, & \delta_b B &= M \omega \Lambda, \\ \delta_b \omega &= 0, & \delta_b \omega^\star &= -\frac{1}{\chi}(\eta_\mu A^\mu + \chi M B) \Lambda,\end{aligned}\quad (3.13)$$

with Λ representing a global, infinitesimal anticommuting parameter.

The path integral (or generating functional) for this gauge-invariant theory is:

$$Z_{ST} \equiv \int D\phi e^{iS_{ST}[\phi]} \quad (3.14)$$

where ϕ denotes the full collection of fields. All correlation functions (Green's functions) follow from differentiating Z_{ST} with respect to appropriate sources.

A. Connecting Proca and Stueckelberg Frameworks

In this part, we demonstrate how Stueckelberg's framework relates to Proca's formulation for massive vector bosons.

We begin by linearizing the Stueckelberg effective action 3.12 via the introduction of an auxiliary field B akin to the Nakanishi-Lautrup field:

$$\begin{aligned}S_{ST} &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left(A_\mu - \frac{1}{M} \eta_\mu B \right)^2 \right. \\ &\quad \left. + \frac{\chi}{2} \mathcal{B}^2 - \mathcal{B}(\eta_\mu A^\mu + \chi M B) - \omega^\star (\eta_\mu \partial^\mu + \chi M^2) \omega \right],\end{aligned}\quad (3.15)$$

which is invariant under the off-shell nilpotent BRST transformations:

$$\delta_b A_\mu = \partial_\mu \omega \Lambda, \quad \delta_b B = M \omega \Lambda, \quad \delta_b \omega = 0, \quad \delta_b \omega^\star = B \Lambda, \quad \delta_b B = 0. \quad (3.16)$$

Correspondingly, the finite field-dependent BRST (FFBRST) transformation takes the form:

$$\delta_b A_\mu = \partial_\mu \omega \Theta[\phi], \quad \delta_b B = M \omega \Theta[\phi], \quad \delta_b \omega = 0, \quad \delta_b \omega^\star = B \Theta[\phi], \quad \delta_b B = 0, \quad (3.17)$$

where Θ is an arbitrary, finite, field-dependent anticommuting parameter.

To establish the link, we select Θ as

$$\Theta' = i\gamma \int d^4x \left[\omega^\star \left(\chi M B - \frac{\chi}{2} \mathcal{B} + \eta_\mu A^\mu \right) \right], \quad (3.18)$$

obtained through Eq. (2.3), with γ an arbitrary constant.

Applying Eq. (2.6), the infinitesimal variation in the Jacobian reads

$$\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \left[\mathcal{B} \left(\chi M B - \frac{\chi}{2} \mathcal{B} + \eta_\mu A^\mu \right) \right], \quad (3.19)$$

where the antighost field equation of motion, $(\partial^2 + \chi M^2)\omega = 0$, is used.

We now propose an ansatz for S_1 as

$$S_1 = \int d^4x [\xi_1(\kappa)\mathcal{B}^2 + \xi_2(\kappa)\mathcal{B}\eta_\mu A^\mu + \xi_3(\kappa)\chi M\mathcal{B}B] \quad (3.20)$$

where the parameters $\xi_i(\kappa)$ satisfy $\xi_i(0) = 0$ initially.

Differentiating S_1 with respect to κ gives

$$\frac{dS_1}{d\kappa} = \int d^4x [\mathcal{B}^2 \xi'_1 + \mathcal{B}\eta_\mu A^\mu \xi'_2 + \chi M\mathcal{B}B \xi'_3], \quad (3.21)$$

where the prime denotes differentiation with respect to κ .

The Jacobian can be expressed as e^{S_1} if the condition in Eq. (2.8) holds:

$$\int d^4x e^{i(S_{ST}+S_1)} \left[i\mathcal{B}^2 \left(\xi'_1 + \frac{\gamma\chi}{2} \right) + i\mathcal{B}\eta_\mu A^\mu (\xi'_2 - \gamma) + i\chi M\mathcal{B}B (\xi'_3 - \gamma) \right] = 0. \quad (3.22)$$

Equating coefficients yields the system

$$\xi'_1 + \frac{\gamma\chi}{2} = 0, \quad \xi'_2 - \gamma = 0, \quad \xi'_3 - \gamma = 0. \quad (3.23)$$

Choosing $\gamma = 1$ without loss of generality, the solutions satisfying the initial values are

$$\xi_1 = -\frac{\chi}{2}\kappa, \quad \xi_2 = \kappa, \quad \xi_3 = \kappa. \quad (3.24)$$

Adding $S_1(\kappa = 1)$ to S_{ST} results in the transformed action

$$S_{ST} + S_1 = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 \left(A_\mu - \frac{1}{M}\eta_\mu B \right)^2 - \omega^* \left(\eta_\mu \partial^\mu + \chi M^2 \right) \omega \right]. \quad (3.25)$$

The generating functional then transforms as

$$Z_{\text{Final}} = \int DA_\mu DBD\omega D\omega^* e^{i(S_{ST}+S_1)} \quad (3.26)$$

Integrating out B , ω , and ω^* fields removes the gauge volume divergence, reducing to the Proca generating functional up to normalization:

$$Z_{\text{Final}} = \int DA_\mu e^{i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A_\mu A^\mu \right]} = Z_P. \quad (3.27)$$

Note that this action is free from divergences since the mass term breaks gauge invariance.

Therefore,

$$Z_{ST} \equiv \int D\phi e^{iS_{ST}} \xrightarrow{\text{FFBRST}} Z_P \equiv \int DA_\mu e^{iS_P} \quad (3.28)$$

Hence, by constructing a suitable finite field-dependent parameter as in Eq. (3.17), we establish that the Stueckelberg generating functional is linked to the Proca generating functional through the FFBRST transformation, implying that Green's functions of these theories are connected within the FFBRST framework.

IV. CONCLUSIONS

The Stueckelberg framework for the massive spin-1 field and the gauge-invariant description of the self-dual chiral boson are examples of first-class theories. Conversely, the Proca model for the massive spin-1 field and the gauge-variant formulation of the self-dual chiral boson exhibit second-class constraints. We have demonstrated that the generalized BRST transformation, characterized by a finite, field-dependent parameter, establishes a connection between the generating functionals of second-class and first-class theories.

Under this FFBRST transformation, the path integral measure is not invariant, and its Jacobian plays a crucial role in relating these two types of theories. This Jacobian effectively cancels the additional terms present in the first-class formulation. Our approach circumvents the complexity involved in computing nonlocal, field-dependent Dirac brackets for second-class theories by instead evaluating the Jacobian corresponding to the FFBRST transformation.

Our findings are illustrated by two explicit examples, where the generating functional of the Stueckelberg theory is linked to that of the Proca model. This method can be extended to connect generating functionals of any first-class theories (such as non-Abelian gauge theories) with second-class theories, provided suitable finite parameters are constructed.

References

- [1] P. A. M. Dirac, *Lectures on Quantum Mechanics*, (Yeshiva Univ. Press, New York, 1964).
- [2] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. **B 55**, (1975) 224.
- [3] C. Becchi, A. Rouet and R. Stora, Annals Phys. **98**, (1976) 287.
- [4] I. V. Tyutin, *LEBEDEV* **39** (1975).
- [5] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**, (1979) 1.
- [6] A. Restuccia and J. Stephany, Phys. Lett. **B 305**, (1993) 348.
- [7] I. A. Batalin and E. S. Fradkin, Nucl. Phys. **B 279**, (1987) 514; Phys. Lett. **B 180**, (1986) 157.
- [8] I. A. Batalin and I. V. Tyutin, Int. J. Mod. Phys. **A 6**, (1991) 3255.
- [9] T. Fujiwara, Y. Igarashi and J. Kubo, Nucl. Phys. **B 341**, (1990) 695.
- [10] Y. W. Kim, S. K. Kim, W. T. Kim, Y. J. Park, K. Y. Kim and Y. Kim Phys. Rev. **D 46**, (1992) 4574.
- [11] L. D. Faddeev and S. L. Shatashvili, Phys. Lett. **B 167**, (1986) 225.
- [12] J. Wess and B. Zumino, Phys. Lett. **B 37**, (1971) 95.
- [13] A. Proca, J. de Phys. et le Radium **7**, (1936) 347.
- [14] R. J. Glauber, Prog. Theor. Phys. **9**, (1953) 295.
- [15] A. Lahiri, ArXiv [hep-th] 9301060 (1993).
- [16] A. Lahiri, Phys. Rev. **D 55**, (1997) 5045.

- [17] A. Lahiri, Phys. Rev. **D 63**, (2001) 105002.
- [18] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, (1938) 225.
- [19] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, (1938) 299.
- [20] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, (1938) 312.
- [21] C. Marshall and P. Ramond, Nucl. Phys. **B 85**, (1975) 375.
- [22] P. Ramond, Prog. Theor. Phys. Suppl. **86**, (1986) 126.
- [23] G. Aldazabal, L. E. Ibanez and F. Quevedo, JHEP **02**, (2000) 015.
- [24] S. D. Joglekar and B. P. Mandal, Phys. Rev. **D 51**, (1995) 1919.
- [25] R. Banerjee and B. P. Mandal, Phys. Lett. **B 27**, (2000) 488.
- [26] S. D. Joglekar and B. P. Mandal, Int. J. Mod. Phys. **A 17**, (2002) 1279 .
- [27] S. Upadhyay, S. K. Rai and B. P. Mandal, J. Math. Phys. **52**, (2011) 022301.
- [28] S. Upadhyay and B. P. Mandal, Eur. Phys. Lett. **93**, (2011) 31001.
- [29] S. Upadhyay and B. P. Mandal, Mod. Phys. Lett. **A 40**, (2010) 3347.
- [30] B. P. Mandal, S. K. Rai and S. Upadhyay, Eur. Phys. Lett. **92**, (2010) 21001.
- [31] S. Upadhyay and B. P. Mandal, Eur. Phys. J. **C 72**, (2012) 2065.
- [32] S. Upadhyay and B. P. Mandal, AIP Conf. Proc. **1444**, 213 (2012).