Class of lattices with two atoms

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Abstract:

For finite lattice L with exactly two atoms there exists a unique ideal B. In this article studied structure of ideal B and proved annihilator of ideal B is zero ideal. Shown that for proper ideal B there exists a distinct dual ideal of L. Further introduced a class of particular two atoms lattices.

Index Terms: Lattice, atom, isomorphic, unique ideal, annihilator *(Keywords)*

1. INTRODUCTION

Associating a graph to an algebraic structure is the interest of researcher in this subject of research and has attracted considerable attention in this area. To find the relationship between algebra and graph theory and applications of each other. The idea of zero divisor graph of commutative ring R with identity was introduced by I Beck [4] in 1988. He defined $T_0(R)$, zero divisor graph of ring R with identity, whose vertices are elements of R and two vertices x and y are adjacent if and only if xy=0. Coloring of $T_0(R)$ was his concern. Beck conjectured that $\chi(R) = \omega(R)$; $\chi(R)$ is the chromatic number and $\omega(R)$ clique number of $T_0(R)$. This investigation was continued by Anderson and Naseer in [1]. They gave counter example of above conjecture. Anderson and Livingston in [2] associated a zero divisor graph to a commutative ring R with identity by different method. They defined T (R) ; zero divisor graph of ring R with identity, whose vertices are non-zero zero divisors of R and two distinct vertices x and y are adjacent if and only if xy=0.

Many authors have studied zero divisor graphs of rings and other graphs associated to other algebraic structures see [3, 5, 7, 10]. Considering zero divisors and their annihilators, annihilating ideal graphs of different structures have studied in [11, 12, 13]. For lattices see [8, 9].

In [6] Deore and Tayade studied the zero divisor graphs of lattices and its unique ideals. Study of zero divisor graph gives idea to introduced a unique ideal related to number of atoms in the lattice. Aim was to expose relation between graph theory and lattice theory.

In this article finite lattices of two atoms and their unique ideals are studied. For proper unique ideal there exists a distinct dual ideal. Each of the unique ideal is annihilated by zero ideal only. Further introduced a class of such lattices. The class contains distributive, non-distributive, modular, non-modular lattices.

In this paper all lattices are finite, atomic, with two atoms and some conditions, if not specified.

2. PRILIMINARIES

We recall some definitions and basic results.

Definition 2.1. A poset (L, $\leq$) is said to form a lattice if Sup{a, b} and Inf{a, b } exists in L, for any a, b$\in$L.In this case we write Inf {a, b }=$a\wedge b$, Sup{a, b}=$a\vee b$.

A lattice with finite number of elements is finite lattice.

Definition 2.2. A lattice is an algebra (L, $\wedge$, $\vee$) if the following conditions holds.

a) $a\wedge a=a$, $a\vee a=a$.

b) $a\wedge b= b\wedge a$, $a\vee b=b\vee a$.

c) $a\wedge(b\wedge c)= (a\wedge b)\wedge c$; $a\vee (b\vee c)= (a\vee b) \vee c$.

d) $a\vee(a\wedge b) =a$, $a\wedge(a\vee b)=a$

Theorem 2.1. Let **(**L,$\wedge$,$\vee$) be a lattice. Define an order $\leq$ as below.

For a, b in L, set $a\leq b$ if and only if $a\wedge b=a$, then **(**L, $\leq$) is an partially ordered set and any two of elements have least upper bound (lub) and greatest lower bound (glb).

Conversely, let P be an partially ordered set with glb{a, b} ,lub {a, b} $\in$P. Define

$a\wedge b=$ glb{a, b} and $a\vee b=$ lub{a,b} then (P, $\wedge$, $\vee$) is lattice.

Let S be non empty subset of lattice L. Then (S,$\wedge$,$\vee$) is called sub lattice of lattice L, if $a\wedge b$, $a\vee b\in S$ for a, $b\in S$.

Let L be a lattice and a, b $\in$L, with $a\leq b$ be any two elements. The closed interval is denoted by [a, b] and defined as [a, b] $=\{x\in L \mid a\leq x\leq b\}$. The interval [a, b] is the sub lattice of L. For x $\in$[a, b] be any element, if their exist $y\in L$ such that $x\wedge y=y\wedge x=a$, $x\vee y=y\vee x=b$ then y is a complement of x relative to [a, b].

A lattice L is said be a bounded if there exists elements, say 0 and 1 in L such that $0\leq x\leq 1$, for all $x\in L$. 0 is least and 1 is greatest element of L.

For $x\in L$, if there exists $y\in L$ such that $x\wedge y=0$, $x\vee y=1$, then y is said to be complement of x. If each element of L has complement the lattice is said to be complemented lattice. If each element of a bounded lattice has a unique complement, the lattice is said to be uniquely complemented.

If b, c are elements in L such that $b < x < c$ for no x in L then c is cover of b.

An element ($0\neq$) $a\in L$ is said to be atom of lattice L if a covers 0. Set of atoms of lattice L is denoted by A (L). For every non zero element $x\in L$, if their exists an atom a, such that $a\leq x$ then the lattice L is said to be atomic lattice. A finite lattice with least element is atomic.

In this article lattices under considerations are finite, atomic with two atoms.

A lattice L is to be distributive if $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ for all $a,b,c\in L$.

A lattice L is called modular if for all $a\geq b$, $a\wedge(b\vee c)=b\vee(a\wedge c)$; a, b, $c\in L$.

A non empty sub set I of L is said to be ideal of lattice L, if $a\vee b\in I$ and $k\wedge a\in I$, for a, $b\in I$, $k\in L$.

Let L be a lattice. For $x\in L$, $(x] =\{ y\in L \mid y\leq x \}$ is principal ideal generated by element x. Let A= {a_1, a_2, $a_{3,\ldots}$, an } be non empty subset of L. Let I**=**{$y\in L \mid y \leq a_1\vee a_2\vee a_3\vee_{\ldots}\vee a_n$ } is ideal of lattice L.

A properly contained ideal I of lattice L is called prime ideal if whenever $a\wedge b\in I$ then $a\in I$ or $b\in I$.

Let L be a lattice. An element x $\in$L is said to be zero divisor if their exist ($0\neq$) $y\in L$ such that $x\wedge y=0$. The set of all zero divisors in lattice L is denoted by Z (L). The set of non-zero zero divisors is Z (L)*= Z (L)/ {0}.

3. THEOREMS & RESULTS

Theorem .3.1. ([6], Theorem 2) Let L be a non diamond lattice with the least element 0 and having a finite set of atoms. Then there exists a finite, distributive and uniquely complemented ideal B of L such that A(L)=A(B).

Corollary .3.1. Let L be a lattice and B be unique ideal of L. Then B is principal ideal.

Proof. Let L be a finite lattice. Its every ideal is principal ideal hence B is principal ideal.

Corollary .3.2. ([6], Corollary2) Let L be a lattice with least element 0 and has two atoms and B be unique ideal of L. Then

(a) Z (B)=B–{0, 1_J}, 1_J is the greatest element of ideal B.

(b) Z (B)$\subseteq$Z(L).

Theorem 3.2. Let L be a finite lattice and A (L) = {a, b} be its set of atoms. Then no element of subset $[a\vee b]\uparrow$ is element of Z (L), where $[a\vee b]\uparrow=\{x\in L \mid x\geq a\vee b\}$.

Proof: We have $a, b \leq a \vee b \leq x$ implies, $x \wedge a = a \neq 0$, $x \wedge b = b \neq 0$.

Example 3.1. If L be a finite lattice and B be it's unique ideal then B may or may not be prime ideal.

Let L be a lattice. For $x\in L$, the annihilator of x, denoted by Ann(x), is defined to be $\{ y\in L \mid y\wedge x=0\}$.

Theorem 3.3. If L be a finite lattice and B be it's unique ideal then Ann (B)= {0}.

Proof: Suppose $0\neq x\in$ Ann (B) then $x\wedge y=0$ for all y belongs to B. Since lattice is atomic, $a \leq x$, $a \wedge x=a$. is impossible

Next theorem gives existence of dual ideal for a proper unique ideal. A non empty subset F of lattice L is said to be dual ideal if $a\wedge b\in F$, $a\vee k\in F$, for $a, b\in F$ and $k\in L$.

Theorem 3.4. Let L be a finite lattice. If unique idea B is proper ideal then there exists a dual ideal F such that $B\cap F=\phi$.

Proof: Let B be proper unique ideal of lattice L. Then $L-B \neq \phi$. For $x\in L-B$, let $F= [x)=\{k\in L \mid k\geq x \}$. Then F is dual ideal of L and $B\cap F=\phi$.

Theorem 3.5. Let L be a finite lattice. If unique idea B is proper ideal and there exists dual ideals F and G such that $B\cap F=\phi$ and $B\cap G=\phi$ then $F\cap G \neq \phi$.

A lattice L is said to be Boolean if it is distributive and uniquely complemented.

Theorem 3.6 .Let L be a finite Boolean lattice with two atoms and B be its unique ideal then B=L.

Let $(L, \leq)$ and $(F, \leq`)$ be two lattices. A one one on to map $g: L\rightarrow F$ is called an lattice isomorphism if $x\leq y \Leftrightarrow g(x)\leq` g(y)$ and we write it as $L\cong F$.

Theorem 3.7. Let L and F are lattices. If $L\cong F$ then $B\cong C$;

where B, C are their unique ideals of L and F respectively.

Example 3.2. If $B\cong C$; then L and F may not be isomorphic. from figure 1 & 2.

4. CLASS OF SOME LATTICES

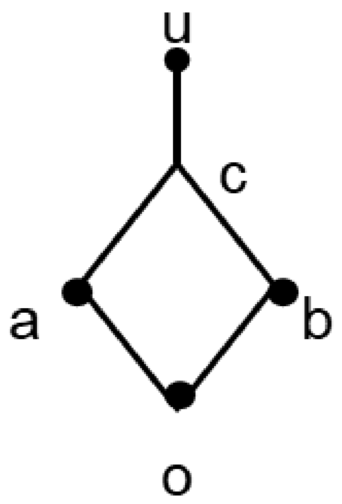

Fig. 1

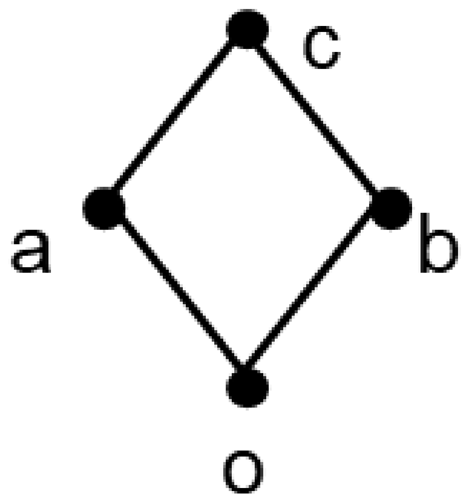

Fig. 2

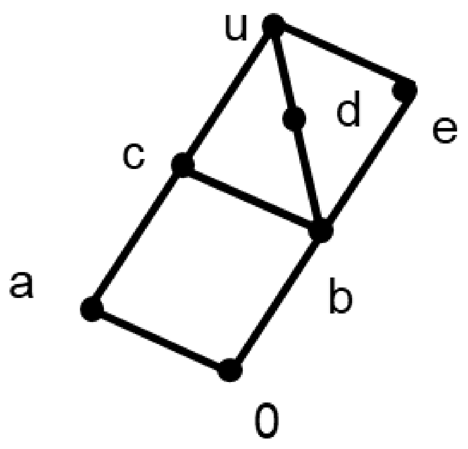

Fig. 3

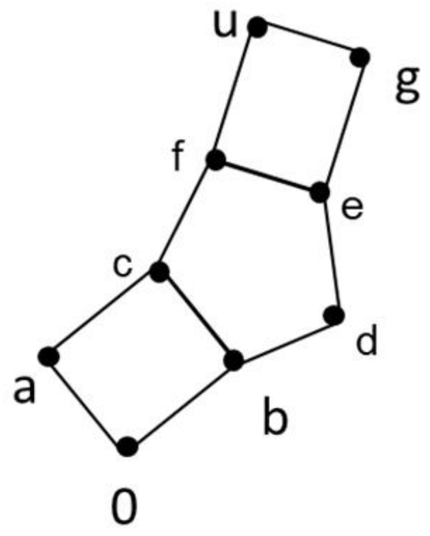

Fig. 4

5. CONCLUSIONS

In this article shown that for proper unique ideal there exists a dual ideal and both are distinct. Unique ideal has no nonzero annihilator ideal.

Above figures represents some finite lattices with two atoms. This class contains distributive, non-distributive, non-complemented, modular, non-modular and Boolean lattices**.** All such lattices constitute a new class.

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