Relating covariant and non-covariant gauges of Stueckelberg theory

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Abstract

This paper explores the generalized BRST symmetry in the framework of finite, field-dependent transformations. The formulation provides a systematic method to connect generating functionals in different gauge choices without spoiling nilpotency or consistency of the quantization scheme. The study applies this approach to the Stueckelberg theory and constructs an explicit mapping between covariant and axial gauges. The Jacobian contribution from the path-integral measure appears as a local functional and establishes the equivalence of the two gauges at the level of Green's functions. The analysis presents a unifying perspective on gauge independence and highlights the utility of generalized BRST transformations in noncovariant quantization.

Keywords: BRST symmetry; Generalized BRST transformations; Gauge-fixing conditions.

I. INTRODUCTION

The Hamiltonian quantization of systems constrained by Dirac's method has long been regarded as a cornerstone in understanding second-class constraints [1]. Within this approach, the introduction of Dirac brackets provides a consistent procedure for eliminating redundant degrees of freedom; however, their field dependence and inherent nonlocality often obscure the canonical structure. In addition, the operator ordering ambiguities that naturally arise in this setting create technical obstacles in the construction of conjugate variables and complicate the algebraic consistency of the quantized theory. By contrast, systems with first-class constraints enjoy a manifestly gauge-invariant quantization, elegantly formulated within the BRST framework [2–5], where the nilpotent BRST charge ensures both unitarity and consistency at the quantum level. To reconcile the disparity between first- and second-class constrained systems, a widely adopted strategy consists in embedding the second-class theory into an enlarged phase space wherein the constraints are effectively converted into first-class ones [2–6]. This embedding method, pioneered and systematized by Batalin, Fradkin, and Tyutin [7, 8], has been successfully applied across a variety of physical models, thereby establishing its robustness and generality [9–12].

A paradigmatic example of such a system is the Proca model in four dimensions, which describes a massive spin-one vector field. The explicit mass term breaks gauge invariance and renders the theory second class [13–17]. To overcome this difficulty, Stueckelberg introduced an auxiliary scalar field to restore gauge invariance, effectively elevating the Proca model to a first-class system [18–20]. This reformulation not only resolves the constraint issue but also provides a fertile framework with far-reaching implications for gauge field theories and string theory constructions [21–23]. The Stueckelberg mechanism thus stands as a key prototype where second-class systems can be systematically reformulated into first-class ones without altering their physical content.

In this work, we investigate the generalized structure of BRST symmetry by extending it to finite, field-dependent transformations. We construct these transformations systematically by introducing a continuous interpolation parameter and by allowing the BRST parameter to depend explicitly on the fields of the theory. We show that the nilpotency of the BRST operator remains intact under this generalization, while the functional measure of the path integral acquires a nontrivial Jacobian. We evaluate this Jacobian and express it as the exponential of a local functional of the fields, which effectively modifies the action without altering the physical content of the theory. We apply this generalized framework to the Stueckelberg theory, which represents a gauge-invariant formulation of a massive vector field. We consider both covariant and axial gauge fixings and construct an explicit mapping between the two by means of finite field-dependent BRST transformations. We analyze the role of the Jacobian in connecting the generating functionals in the two gauges and demonstrate that the Green's functions remain gauge equivalent. This analysis establishes that the generalized BRST symmetry not only preserves the algebraic structure of the usual BRST formulation but also provides a powerful tool to interpolate consistently between covariant and noncovariant quantization schemes. Through this investigation, we present a unifying perspective on gauge equivalence, highlight the broader applicability of generalized BRST transformations, and lay the groundwork for their potential extension to other models beyond the Stueckelberg framework.

The paper is organized as follows. Section II presents a brief review of the conventional BRST symmetry and its role in gauge theories. Section III introduces the generalized BRST transformations and discusses their construction, algebraic properties, and the associated Jacobian of the path-integral measure. In Section IV, the formalism is applied to the Stueckelberg theory, where we explicitly construct the mapping between covariant and non-covariant gauges. Section V establishes the equivalence of the corresponding generating functionals and Green's functions. Section VI is devoted to a discussion of the results and their implications. Finally, Section VII summarizes the main conclusions and outlines possible directions for future work.

II. GENERALIZED BRST TRANSFORMATION

In this part, we reconsider the framework of conventional BRST symmetry and explore its possible extensions. The key observation is that the essential algebraic properties of BRST variations remain valid regardless of whether the anticommuting parameter θ_b is infinitesimal or finite, field dependent or independent, provided that it does not involve any explicit space-time variation. This property provides sufficient flexibility to reformulate the BRST transformations with θ_b promoted to a finite, field-dependent parameter, while still retaining nilpotency.

To accomplish this extension, we begin by making the infinitesimal BRST parameter field dependent. For technical clarity, we introduce an auxiliary interpolation parameter κ with $0 \leq \kappa \leq 1$. All dynamical variables are then assumed to acquire κ-dependence, denoted $\phi(x, \kappa)$, such that the boundary conditions are $\phi(x, \kappa = 0) = \phi(x)$ and $\phi(x, \kappa = 1) = \phi'(x)$, where $\phi'(x)$ denotes the fully transformed configuration.

The generic infinitesimal form of such a transformation can be expressed as

$$d\phi(x, \kappa) = \delta_b[\phi(x, \kappa)]\, \Theta^{'}[\phi(x, \kappa)]\, d\kappa, \tag{2.1}$$

where $\Theta^{'}[\phi(x, \kappa)]\, d\kappa$ plays the role of a κ-dependent but infinitesimal parameter. By integrating these successive transformations across the interpolation interval, one arrives at the finite field-dependent BRST transformation,

$$\phi^{'}(x) = \phi(x) + \delta_b[\phi(x)]\, \Theta[\phi(x)], \tag{2.2}$$

with

$$\Theta[\phi(x)] = \int_0^1 d\kappa'\, \Theta'[\phi(x, \kappa')] \tag{2.3}$$

Although the generalized BRST transformations preserve nilpotency and remain a symmetry of the effective action, they fail to leave the path-integral measure invariant due to the finiteness of the parameter. This lack of invariance is captured by the Jacobian of the functional measure, which for a suitable choice of $\Theta[\phi(x)]$ can be written as

$$D\phi = J(\kappa)\, D\phi(\kappa) = J(\kappa + d\kappa)\, D\phi(\kappa + d\kappa). \tag{2.4}$$

Since the transition $\phi(\kappa) \longrightarrow \phi(\kappa + d\kappa)$ is infinitesimal, the ratio of Jacobians satisfies

$$\frac{J(\kappa)}{J(\kappa + d\kappa)} = \Sigma_\phi \pm \frac{\delta\phi(x, \kappa)}{\delta\phi(x, \kappa + d\kappa)} \tag{2.5}$$

where the summation Σ_ϕ extends over all dynamical variables, and the sign is chosen according to whether ϕ is bosonic (+) or fermionic (−).

The infinitesimal variation of $J(\kappa)$ then follows as

$$\frac{1}{J}\frac{dJ}{d\kappa} = -\int d^4x \left[\pm\delta_b\phi(x, \kappa)\, \frac{\partial\Theta'[\phi(x, \kappa)]}{\partial\phi(x, \kappa)}\right] \tag{2.6}$$

Within the functional integral, the Jacobian J(κ) may be consistently replaced by an exponential of a local functional $S_1[\phi(x, \kappa)]$, i.e.

$$J(\kappa) \longrightarrow \exp[iS_1[\phi(x, \kappa)]], \tag{2.7}$$

Whenever the constraint

$$\int D\phi(x) \left[\frac{1}{J}\frac{dJ}{d\kappa} - i\frac{dS_1[\phi(x, \kappa)]}{d\kappa}\right] \exp\{i[S_{eff} + S_1]\} = 0 \tag{2.8}$$

is satisfied [24].

By a judicious choice of the parameter $\Theta[\phi(x)]$, one can therefore reinterpret the generalized BRST-transformed generating functional

$$Z' = \int D\phi \exp\{i(S_{eff} + S_1)\}$$

as corresponding to another effective quantum theory linked to the original by finite, field-dependent BRST symmetry.

III. STUECKELBERG THEORY

In order to address certain limitations inherent in the Proca framework, Stueckelberg proposed an extended form of the Lagrangian density, expressed as

$$S_{ST} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left(A_\mu - \frac{1}{M} \partial_\mu B \right)^2 \right] \tag{3.1}$$

by supplementing the theory with an additional real scalar field B.

The above functional remains invariant under a local gauge variation given by

$$\delta_g A_\mu(x) = \partial_\mu \lambda(x), \tag{3.2}$$

$$\delta_g B(x) = M\, \lambda(x), \tag{3.3}$$

where λ denotes the gauge function. For quantization, it becomes necessary to fix a gauge. Adopting the 't Hooft gauge condition,

$$\mathcal{L}_{gf} = -\frac{1}{2\chi} \left(\partial^\mu A_\mu + \chi M B \right)^2 \tag{3.4}$$

with χ representing an arbitrary gauge parameter, ensures that the propagators display well-controlled ultraviolet behavior. Consequently, the renormalizability of the theory is maintained.

We now focus on the BRST symmetry in the Stueckelberg framework. Introducing the ghost field (ω) along with the antighost ($\omega^\star$), the effective Lagrangian can be expressed as

$$\begin{aligned} S^L_{ST} = \int d^4x \Big[&- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \Big(A_\mu - \frac{1}{M} \partial_\mu B \Big)^2 \\ &- \frac{1}{2\chi} (\partial^\mu A_\mu + \chi M B)^2 - \omega^\star (\partial^2 + \chi M^2) \omega \Big]. \end{aligned} \tag{3.5}$$

This action is symmetric under the following BRST operations:

$$\begin{aligned} \delta_b A_\mu = \partial_\mu \omega\, \theta_b, &\quad \delta_b B = M \omega\, \theta_b, \\ \delta_b \omega = 0, &\quad \delta_b \omega^\star = -\frac{1}{\chi} (\partial_\mu A^\mu + \chi M B)\, \theta_b \end{aligned} \tag{3.6}$$

where θ_b denotes a global, anticommuting, infinitesimal parameter.

In a non-covariant gauge, the modified effective action becomes

$$S_{ST}^{A} = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}\Big(A_\mu - \frac{1}{M}\partial_\mu B\Big)^2 - \frac{1}{2\chi}(\eta^\mu A_\mu + \chi M B)^2 - \omega^\star(\eta\cdot\partial + \chi M^2)\omega\Big] \tag{3.7}$$

This expression also remains invariant under the BRST transformations defined in (3.6).

Finally, the generating functional in the covariant gauge is written as

$$Z^{L}{}_{ST} \equiv \int D\phi \, \exp(iS_{ST}{}^{L}\,[\phi]), \tag{3.8}$$

where ϕ collectively denotes all fields appearing in the formalism. Green's functions associated with the theory can be systematically obtained from Z^L.

IV. RELATING STUECKELBERG IN COVARIANT AND NON-COVARIANT GAUGES

We begin by analyzing the linearized version of the Stueckelberg effective action (3.5). To handle the gauge-fixing consistently, an auxiliary Nakanishi–Lautrup field B is introduced. With this inclusion, the action takes the form

$$S_{ST}^{L} = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2\Big(A_\mu - \frac{1}{M}\partial_\mu B\Big)^2 + \frac{\chi}{2}\mathcal{B}^2 - \mathcal{B}(\partial_\mu A^\mu + \chi M B) - \omega^\star(\partial^2 + \chi M^2)\omega\Big], \tag{4.1}$$

where the first term describes the gauge field kinetic energy, the second term encodes the Stueckelberg mass mechanism, the third term is the quadratic contribution of the auxiliary field, the fourth implements gauge fixing, and the last represents the ghost sector.

This action is invariant under the following nilpotent BRST symmetry (off-shell):

$$\delta_b A_\mu = \partial_\mu \omega\, \theta_b\,,\;\; \delta_b B = M\,\omega\,\theta_b\,,\;\; \delta_b \omega = 0\,,\;\; \delta_b \omega^\star = \boldsymbol{B}\,\theta_b\,,\;\; \delta_b \boldsymbol{B} = 0. \tag{4.2}$$

Here, θ_b is a constant Grassmann parameter. The transformation mixes the physical fields (A_μ, B) with the ghost sector (ω, $\omega^\star$) and ensures consistency of quantization by preserving gauge invariance at the quantum level.

When the same structure is reformulated in the axial-type gauge, the action modifies to

$$S_{ST}^{A} = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2\Big(A_\mu - \frac{1}{M}\partial_\mu B\Big)^2 + \frac{\chi}{2}\mathcal{B}^2 - \mathcal{B}(\eta_\mu A^\mu + \chi M B) - \omega^\star(\eta\cdot\partial + \chi M^2)\omega\Big], \tag{4.3}$$

where η_μ is a fixed direction in spacetime that enforces the axial gauge condition.

To interpolate between these two gauge choices, a generalized BRST transformation is defined:

$$\delta_b A_\mu = \partial_\mu \omega \, \Theta[\phi], \quad \delta_b B = M \, \omega \, \Theta[\phi], \quad \delta_b \omega = 0, \quad \delta_b \omega^\star = \boldsymbol{B} \, \Theta[\phi], \quad \delta_b \boldsymbol{B} = 0. \tag{4.4}$$

The key modification is the appearance of $\Theta[\phi]$, a finite field-dependent Grassmann parameter. Unlike the constant θ_b, this parameter depends on the dynamical fields, allowing a continuous mapping between different gauges.

For explicit construction, Θ is chosen as

$$\Theta' = i\gamma \int d^4x \, [\omega^\star(\partial_\mu A^\mu - \eta_\mu A^\mu)] \,, \tag{4.5}$$

with γ an arbitrary constant. This specific form ensures that the covariant and axial gauges can be connected via a finite transformation.

Next, we analyze how the functional measure changes under such a transformation. From Eq. (2.6), the variation of the Jacobian is obtained as

$$\frac{1}{J}\frac{dJ}{d\kappa} = i\gamma \int d^4x \Big[\mathcal{B}(\partial_\mu A^\mu - \eta_\mu A^\mu) + \omega^\star(\partial_\mu A^\mu - \eta_\mu A^\mu)\omega\Big] \tag{4.6}$$

This nontrivial Jacobian factor must be compensated by an additional term in the action to preserve the path integral.

We therefore make the ansatz

$$S_1 = \int d^4x \, [\, \xi_1(\kappa)\boldsymbol{B}\partial_\mu A^\mu + \xi_2(\kappa)\boldsymbol{B}\eta_\mu A^\mu + \xi_3(\kappa)\omega^\star\partial_\mu A^\mu \omega + \xi_4(\kappa)\omega^\star\eta_\mu A^\mu \omega \,] \,, \tag{4.7}$$

with $\xi_i(\kappa = 0) = 0$. The ξ_i functions encode the dependence on the interpolation parameter κ.

Differentiating with respect to κ gives

$$\frac{dS_1}{d\kappa} = \int d^4x \Big[\xi_1' \mathcal{B}\partial_\mu A^\mu + \xi_2' \mathcal{B}\eta_\mu A^\mu + \xi_3' \omega^\star \partial_\mu A^\mu \omega + \xi_4' \omega^\star \eta_\mu A^\mu \omega\Big] \tag{4.8}$$

Matching this expression with the Jacobian contribution leads to the consistency conditions

$$\xi_1' - \gamma = 0, \qquad \xi_2' + \gamma = 0, \qquad \xi_3' - \gamma = 0, \qquad \xi_4' + \gamma = 0 \tag{4.9}$$

Solving for $\gamma = 1$ yields

$$\xi_1 = 1, \qquad \xi_2 = -1, \qquad \xi_3 = 1, \qquad \xi_4 = -1, \tag{4.10}$$

which satisfy the required initial conditions.

Substituting these values, the modified action becomes

$$S^L_{ST} + S_1 = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2\left(A_\mu - \frac{1}{M}\partial_\mu B\right)^2 + \frac{\chi}{2}\mathcal{B}^2 - \mathcal{B}(\eta_\mu A^\mu + \chi M B) - \omega^\star(\eta\cdot\partial + \chi M^2)\omega\Big], \tag{4.11}$$

which is precisely the axial gauge action.

Finally, the generating functional under this finite field-dependent BRST transformation transforms as

$$Z' = \int DA_\mu DBD\omega D\omega^\star \exp\left\{ i\int d^4x \big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2\left(A_\mu - \frac{1}{M}\partial_\mu B\right)^2 + \frac{\chi}{2}\mathcal{B}^2 - \mathcal{B}(\eta_\mu A^\mu + \chi M B) - \omega^\star(\eta\cdot\partial + \chi M^2)\omega\big] \right\} = Z^A, \tag{4.12}$$

Implying

$$Z^L\left(\int DA_\mu DBD\omega D\omega^\star e^{iS^L_{ST}}\right) \overset{Generalized\ BRST}{- - - \longrightarrow} Z^A\left(\int DA_\mu DBD\omega D\omega^\star e^{iS^A_{ST}}\right). \tag{4.13}$$

This demonstrates explicitly that a carefully constructed finite, field-dependent BRST parameter connects the generating functional in covariant gauge to that in axial gauge. Consequently, the corresponding Green's functions in both formulations remain related via the generalized BRST approach, ensuring consistency of the quantization procedure.

V. CONCLUSION

In this work, we have reconsidered the structure of BRST symmetry and have shown that its generalization to finite, field-dependent transformations has provided a powerful tool for relating different gauge-fixed versions of a theory. By introducing an interpolation parameter, we have constructed the finite field-dependent BRST transformations while retaining nilpotency and consistency of the quantization procedure.

We have further applied this generalized framework to the Stueckelberg theory. By selecting a suitable form of the field-dependent parameter, we have established an explicit mapping between the covariant and the axial gauges. The Jacobian of the path-integral measure has been computed, and its contribution has been absorbed into a local functional, thereby ensuring that the generating functionals in different gauges have been related consistently.

Thus, we have demonstrated that the generalized BRST formulation has not only preserved the algebraic structure of the symmetry but has also connected the Green's functions of covariant and non-covariant gauges in a systematic manner. This framework has therefore provided a unifying perspective on gauge equivalence within the quantization of the Stueckelberg model.

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