A Mapping of Different Gauges of Thirring Model Using Generalized BRST Transformation

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Abstract

In this work, we investigate the BRST symmetry structure of the Thirring model in arbitrary dimensions. Starting from the original Thirring Lagrangian, we reformulate it in terms of an auxiliary vector field and subsequently construct a gauge-invariant extension by introducing the Stueckelberg field. The resulting theory admits a local $U(1)$ gauge symmetry, enabling covariant quantization through the inclusion of appropriate gauge-fixing and Faddeev–Popov ghost terms. We demonstrate explicit BRST invariance of the effective action in both axial and Lorentz gauges. Extending the formalism, we derive generalized (finite field-dependent) BRST transformations and analyze their impact on the path integral measure. The associated non-trivial Jacobian is computed explicitly, and we show that it modifies only the BRST-exact sector of the action, thereby preserving the physical content of the theory. As an application, we establish a precise mapping between generating functionals in axial and Lorentz gauges, mediated through the generalized BRST symmetry. This construction highlights the equivalence of different gauge-fixed versions of the Thirring model and provides a unified framework for studying its BRST cohomological structure.

Keywords: BRST transformations; Gauge conditions; Finite field-dependent BRST transformations.

I. INTRODUCTION

The study of quantum field theories (QFTs) has revealed that the vacuum structure is highly sensitive to the number of fermionic flavors. This property has motivated the investigation of various four-fermion interaction models. A prominent example is the Thirring model, which describes fermions interacting through a conserved vector current in three dimensional spacetime. Historically, the Thirring model has been regarded as an effective framework for exploring the phenomenon of dynamical mass generation of fermions [1– 3]. Such mechanisms are central to scenarios of electroweak symmetry breaking beyond the standard model, including technicolor-type models [4] and top-quark condensation approaches [5].

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The role of four-fermion interactions has been revisited in different contexts, ranging from walking technicolor [6] to strong extended technicolor [7]. Particularly, scalar and pseudoscalar interactions in four dimensions acquire a special significance when coupled with gauge interactions, leading to a renormalizable theory [8]. The resulting theory is referred to as the gauged Nambu–Jona-Lasinio (NJL) model [9–11], which shows remarkable similarities in its phase structure to the Gross–Neveu model in three dimensions [13]. Another natural generalization is the gauged Thirring model, which interpolates between these frameworks and, in the strong coupling limit, reproduces a gauge-theoretic reformulation of the original Thirring model [14].

Quantization of such gauge theories requires a systematic procedure, for which the Becchi–Rouet–Stora–Tyutin (BRST) formalism [15–18] provides a consistent tool. It ensures both renormalizability and unitarity. Various aspects of BRST quantization of the Thirring model in dimensions $2 < D < 4$ have already been analyzed [3], where different gauges, such as the Lorentz and axial gauges, were compared. Establishing a connection between distinct gauge choices becomes valuable since certain computations may simplify considerably in one gauge but not in another. This motivates a generalized BRST construction that enables such mappings.

A key refinement of the standard BRST symmetry is obtained by promoting the transformation parameter to a finite, field-dependent functional, producing what is known as generalized BRST transformations [19]. This extension has been successfully applied to numerous gauge systems [20, 21], as well as to gravity-inspired models [22]. For instance, the Gribov ambiguity in Yang–Mills theory [23–25] has been addressed effectively using this framework [26]. Applications also extend to higher-form gauge theories relevant for string theory [27], and to superconformal Chern–Simons–matter theories [28–33]. The validity of the generalized BRST approach has even been established at the quantum level using the Batalin–Vilkovisky (BV) formalism [34–36].

It is well known that four-fermion current–current interactions are nonrenormalizable in four dimensions since the coupling carries negative mass dimension. Nevertheless, appropriate expansions allow certain D-dimensional variants to remain renormalizable [39]. In this work, we focus on a D-dimensional Thirring model that lacks explicit gauge invariance. Introducing auxiliary fields transforms the system into an equivalent vector theory, though still requiring gauge fixing for proper quantization. Specifically, we analyze the Lorentz and axial gauges, writing the corresponding Faddeev–Popov actions.

We then generalize the BRST symmetry by introducing finite, field-dependent parameters. While the classical and gauge-fixed action remains BRST invariant, the path integral measure acquires a nontrivial Jacobian. This Jacobian depends on the chosen field-dependent parameter and induces modifications in the generating functional. Remarkably, with a specific parameter choice, the Jacobian connects the functional in one gauge to that in another. We explicitly demonstrate this phenomenon for the Lorentz and axial gauges.

The rest of the paper is organized as follows: Section II reviews BRST quantization of the Thirring model with general and specific gauges. Section III develops the extended BRST symmetry and computes the

Jacobian for the path integral measure. Section IV discusses the mapping between distinct gauge choices through the generalized symmetry. Finally, Section V concludes the work with a summary of results and possible future directions.

II. THIRRING MODEL AND BRST SYMMETRY

The Thirring model is a well-known quantum field theory of fermions interacting through a current-current coupling. In D-dimensional space-time, its basic Lagrangian density is expressed as

$$\mathcal{L}_{\rm Th} = \bar{\psi}^a\, i\gamma^\mu \partial_\mu \psi^a - m\,\bar{\psi}^a\psi^a - \frac{G}{2N}\,(\bar{\psi}^a\gamma^\mu\psi^a)(\bar{\psi}^b\gamma_\mu\psi^b). \tag{2.1}$$

Here ψ^a denotes a Dirac fermion carrying a flavor index $a = 1,\dots,N$. The matrices γ^μ are the usual generators of the Clifford algebra, obeying

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}\,\mathbf{1}. \tag{2.2}$$

The first term in Eq. (2.1) describes free propagation of the fermions, the second provides a mass contribution, while the final term represents the characteristic four-fermion interaction.

A. Auxiliary Field Formulation

Direct treatment of the quartic fermionic interaction can be complicated. A common trick is to replace the interaction by introducing an auxiliary bosonic field A_μ, which does not correspond to a dynamical gauge boson but instead linearizes the interaction. This reformulation yields the equivalent Lagrangian

$$\mathcal{L}_{\rm Th'} = \bar{\psi}^a i\gamma^\mu\Big(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu\Big)\psi^a - m\bar{\psi}^a\psi^a + \frac{1}{2G}A_\mu A^\mu. \tag{2.3}$$

The equation of motion for A_μ is algebraic; substituting its solution reproduces the original four-fermion form (2.1). Thus, A_μ effectively plays the role of a composite current field rather than a genuine gauge connection. In particular, there is no kinetic term for A_μ, and hence no dynamical Yang–Mills structure.

B. Gauge-Invariant Extension

To construct a gauge-invariant counterpart of the theory, one can introduce a Stueckelberg scalar θ. This additional degree of freedom compensates the lack of gauge symmetry in Eq. (2.3). The modified Lagrangian is then

$$\mathcal{L}_{\rm Th''} = \bar{\psi}^a i\gamma^\mu\Big(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu\Big)\psi^a - m\bar{\psi}^a\psi^a + \frac{1}{2G}\Big(A_\mu - \sqrt{N}\partial_\mu\theta\Big)^2. \tag{2.4}$$

This form is invariant under a local $U(1)$ symmetry with transformation rules

$$\delta\psi_a = (e^{i\phi} - 1)\psi_a \quad , \tag{2.5}$$

$$\delta A_\mu = \sqrt{N}\,\partial_\mu\phi, \tag{2.6}$$

$$\delta\theta = \phi, \tag{2.7}$$

where ϕ is a local parameter interpreted as a would-be Nambu–Goldstone boson. The scalar θ ensures that A_μ effectively acquires a longitudinal component, mimicking the role of a massive vector boson in gauge theories.

C. Gauge Fixing and Ghosts

While Eq. (2.4) enjoys gauge symmetry, quantization requires elimination of redundant degrees of freedom. This is achieved through gauge fixing. The generic procedure introduces an auxiliary field B (of Nakanishi–Lautrup type) and ghost/anti-ghost fields ($C,\bar{C}$). The most general gauge-fixing contribution reads

$$\mathcal{L}_{\rm GF} = B\,\mathcal{F}[A] + \frac{\xi}{2}B^2 + i\bar{C}\,\frac{\delta\mathcal{F}[A]}{\delta A_\mu}\,\partial_\mu C, \tag{2.8}$$

with F[A] specifying the gauge choice and ξ an arbitrary real parameter controlling the gauge condition.

1. *Axial Gauge*

For the axial gauge, the choice F[A] = $\tau^\mu A_\mu$ simplifies the ghost sector considerably, yielding

$$\mathcal{L}^A_{\rm GF} = B\,\tau^\mu A_\mu + \frac{\xi}{2}B^2 + i\bar{C}\,(\tau_\mu\partial^\mu)C. \tag{2.9}$$

The full effective Lagrangian becomes

$$\mathcal{L}^A_{\rm eff} = \bar{\psi}^a i\gamma^\mu\Big(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu\Big)\psi^a - m\bar{\psi}^a\psi^a + \frac{1}{2G}(A_\mu - \sqrt{N}\partial_\mu\theta)^2 + B\tau^\mu A_\mu$$
$$+\frac{\xi}{2}B^2 + i\bar{C}\,\tau_\mu\partial^\mu C. \tag{2.10}$$

This theory is invariant under the nilpotent BRST symmetry:

$$\begin{aligned}
&\delta_b A_\mu = -\partial_\mu C\,\tau, \\
&\delta_b B = 0, \quad \delta_b C = 0, \\
&\delta_b \bar{C} = iB\tau, \quad \delta_b \theta = -\frac{1}{\sqrt{N}} C\tau, \\
&\delta_b \psi^j = \frac{i}{\sqrt{N}} C\psi^j \tau,
\end{aligned} \tag{2.11}$$

where τ is an anticommuting infinitesimal parameter.

2. Lorentz Gauge

Alternatively, if the Lorentz condition $F[A] = \partial_\mu A^\mu$ is chosen, the gauge-fixing term becomes

$$\mathcal{L}^L_{\text{GF}} = B\,(\partial_\mu A^\mu) + \frac{\xi}{2} B^2 + i\bar{C}(\partial_\mu \partial^\mu) C. \tag{2.12}$$

The effective Lagrangian now reads

$$\begin{aligned}
\mathcal{L}^L_{\text{eff}} = &\bar{\psi}^a i\gamma^\mu \Big(\partial_\mu - \frac{i}{\sqrt{N}} A_\mu\Big)\psi^a - m\bar{\psi}^a\psi^a + \frac{1}{2G}(A_\mu - \sqrt{N}\partial_\mu\theta)^2 + B(\partial_\mu A^\mu) \\
&+ \frac{\xi}{2} B^2 + i\bar{C}(\partial_\mu\partial^\mu) C,
\end{aligned} \tag{2.13}$$

and preserves the same BRST invariance as the axial gauge.

III. GENERALIZED BRST TRANSFORMATIONS

The conventional BRST transformation of a field $\phi(x)$ is

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + s_b\phi(x)\tau, \tag{3.1}$$

where s_b denotes the BRST operator. A natural extension is to replace the infinitesimal, constant parameter τ by a finite, field-dependent one. To achieve this, one interpolates

fields with a continuous variable κ:

$$\frac{d}{d\kappa}\phi(x,\kappa) = s_b\phi(x,\kappa)\,\Theta'[\phi(x,\kappa)]. \tag{3.2}$$

After integration, this yields the finite transformation

$$\delta_b\phi(x) = s_b\phi(x)\Theta[\phi(x)], \tag{3.3}$$

with

$$\Theta[\phi] = \Theta'[\phi]\,\frac{e^{f[\phi]} - 1}{f[\phi]}, \qquad f[\phi] = \sum_i \int d^4x\, \frac{\delta\Theta'[\phi]}{\delta\phi_i(x)}\, s_b\phi_i(x). \tag{3.4}$$

These generalized transformations leave the action invariant but alter the measure of the path integral.

A. Jacobian of the Transformation

Consider the generating functional of the theory,

$$Z[0] = \int \mathcal{D}\phi \, e^{iS_{\mathrm{Th}}^{FP}[\phi]} . \tag{3.5}$$

The functional measure changes as

$$D\phi \;\rightarrow\; J[\phi]D\phi, \tag{3.6}$$

where the Jacobian satisfies

$$\frac{1}{J}\frac{dJ}{d\kappa} = -\int d^Dx \sum_\phi \pm s_b\phi(x,\kappa)\,\frac{\delta\Theta'[\phi(x,\kappa)]}{\delta\phi(x,\kappa)} . \tag{3.7}$$

After integration over κ, one finds the explicit result

$$J[\phi] = \exp\left(-\int d^Dx \sum_\phi \pm s_b\phi(x)\,\frac{\delta\Theta'[\phi(x)]}{\delta\phi(x)}\right). \tag{3.8}$$

This Jacobian modifies only BRST-exact pieces of the effective action, and therefore does not influence the physical observables of the model.

IV. RELATION BETWEEN AXIAL AND LORENTZ GAUGES

As an illustrative example, consider the generalized BRST transformation rules with finite parameter $\Theta[\phi]$. Choosing

$$\Theta'[\phi] = \int d^Dx\, \bar{C}\,(\tau_\mu A^\mu - \partial_\mu A^\mu), \tag{4.1}$$

the Jacobian takes the form

$$J[\phi] = \exp\left\{i\int d^Dx\left[-B\tau^\mu A_\mu - i\bar{C}\tau_\mu\partial^\mu C + B\partial^\mu A_\mu + i\bar{C}\partial_\mu\partial^\mu C\right]\right\}. \tag{4.2}$$

This explicitly shows that the generating functional in axial gauge is mapped to that in Lorentz gauge. The contribution of the Jacobian is BRST exact, and hence does not alter the physical content of the model. More generally, by selecting

$$\Theta'[A, \bar{C}] = \int d^Dx\, \bar{C}\,(F_1[A] - F_2[A]), \tag{4.3}$$

one can relate any two arbitrary gauge choices, establishing that different gauges are connected through finite, field-dependent BRST transformations.

V. CONCLUSION

In this work, we have analysed the BRST symmetry of the Thirring model in arbitrary dimensions and have reformulated the theory in its gauge-invariant form by introducing the Stueckelberg field. We have investigated the covariant quantization of the theory by implementing different gauge-fixing conditions and have shown that the effective actions in both axial and Lorentz gauges have remained invariant under BRST transformations. Furthermore, we have extended the BRST symmetry to its generalized version by introducing finite field-dependent parameters, and we have derived the corresponding Jacobian for the functional measure. This Jacobian has been shown to contribute only to the BRST-exact part of the action, so the physical content of the theory has remained unchanged.

Finally, we have demonstrated that the generalized BRST transformation has provided a natural connection between generating functionals in different gauges, explicitly establishing the equivalence between the axial and Lorentz gauges. Thus, we have established that the Thirring model in its BRST framework has admitted a consistent quantization scheme and that the generalized BRST symmetry has served as a powerful tool to relate various gauge-fixed versions of the theory.

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