Stability Analysis of Nonlinear Non-Autonomous Systems

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Abstract: **This paper is concerned with Stability Analysis of Nonlinear Non-autonomous systems by using Lyabunov functions and the time derivative of a Lyabunov functions are indefinite in the Lyabunov stability theorems and it can be achieved with the help of some scalar functions.**

Keywords: **Nonlinear systems, Lyabunov functions, Asymptotic stability.**

1. INRTODUCTION

The most helpful approach for studying of nonlinear systems is the theory introduced in 19th century by the Russian Mathematician Alexandar Mikhailovich Lyabunov. A powerful tool for analyzing stability is the use of Lyabunov functions.

A Lyabunov function is an energy like function that can be used to analyse stability of a system. The Lyabunov indirect method which is also known as the Lyabunov's second method.

The stability of Non- autonomous (time variant) systems are more arduous to handle than autonomous (time invariant) systems. The classical Lyabunov stability theorems are generalized that the time derivative Lyabunov functions are allowed to be indefinite.

Stability of the system can be guaranteed if some scalar function is stable and different stability properties of the scalar function together with different assumptions on the bound of Lyabunov functions, which provides different stability outcomes of the nonlinear system.

The paper is organized as follows. Section 2 deals with the definitions of stability analysis in Control Theory and Section 3 represents the stability theorems by using Lyabunov functions whose time derivative can take indefinite values of nonlinear non-autonomous system and Section 4 concludes the paper.

2. PRELIMINARIES

Consider the nonlinear non-autonomous system

$$\dot{y} = g(t, y(t), v(t)) \tag{1}$$

Where $f: J \times R^n \times R^m \to R^n$ is continuous, locally lipschitz on y for bounded v and $g(t, 0,0) = 0$. The input $v: J \to R^m$ is assumed to be locally essentially bounded.

And for any C^1 function $: J \times R^n \to R$, defined by

$$\dot{U}(t, y)\big|_{(1)} \triangleq \frac{\partial U(t,y)}{\partial t} + \frac{\partial U(t,y)}{\partial y} g(t, y, v) \tag{2}$$

Consider the scalar linear time varying system is given by,

$$\dot{x} = \gamma(t)x(t), \ t \in J \tag{3}$$

Where $x(t): J \to R$ is state variable and $\gamma(t) \in \mathbb{PC}(J, R)$.

Then the state transition matrix for the system $\dot{x} = \gamma(t)x(t)$ is,

$$\varphi(t, t_0) = exp\left(\int_{t_0}^{t} \gamma(s)\, ds\right), \ \forall t_0 \le t \in J \tag{4}$$

DEFINITON: 1

The zero solution of the system $\dot{y} = g\big(t, y(t), v(t)\big)$ is said to be *stable* if for every $\varepsilon > 0$, there exist a number $\delta(t_0, \varepsilon) > 0$ such that $\|y(t_0)\| < \delta(t_0, \varepsilon)$ implies

$$\|y(t)\| < \varepsilon, \qquad \forall t_0 \le t.$$

DEFINITON: 2

The Nonlinear Non-autonomous system $\dot{y} = g\big(t, y(t), v(t)\big)$ is said to be *globally asymptotically stable* if for any $\varepsilon > 0$, there is $\delta(t_0, \varepsilon) > 0$ such that $|y(t_0)| \le \delta(t_0, \varepsilon)$ implies

$$\|y(t)\| < \varepsilon, \quad \forall t_0 \le t \in J$$

and for any $x(t_0) \in R^n$, there holds $\lim_{t\to\infty} |y(t)| = 0$.

DEFINITON: 3

The Nonlinear Non-autonomous system $\dot{y} = g\big(t, y(t), v(t)\big)$ is said to be *globally uniformly asymptotically stable* if there exists a $\sigma \in \mathcal{KL}$ such that, for any $y(t_0) \in R^n$ satisfies

$$|y(t)| \le \sigma(|y(t_0)|, t - t_0), \quad \forall t_0 \le t \in J$$

DEFINITON: 4

The Nonlinear Non-autonomous system $\dot{y} = g\big(t, y(t), v(t)\big)$ is said to be *globally exponentially stable* if there is a $\theta \in \mathcal{N}$ and $\beta > 0$ such that

$$|y(t)| \le \theta(t_0)|y(t_0)|\, exp\big(-\beta(t - t_0)\big), \ \forall t_0 \le t \in J$$

DEFINITON: 5

The Nonlinear Non-autonomous system $\dot{y} = g\big(t, y(t), v(t)\big)$ is said to be *globally uniformly exponentially stable* if it is *globally exponentially stable* with $\theta(t_0)$ independent of t_0.

3. ASYMPTOTIC STABILITY ANALYSIS

In this section we shall provide the theorems for stability analysis such as asymptotic stability and exponential stability.

THEOREM: 1

Assume that there exist two $\mathcal{NK}_\infty$ functions β_i, $i = 1,2$ and C^1 function $U: J \times R^n \to [0, \infty)$ and a scalar function $\gamma(t) \in \mathbb{PC}(J, R)$ such that for all $t \in J$ and $y \in R^n$,

$$\beta_1(t, |y|) \le U(t, y) \le \beta_2(t, |y|),$$

$$\dot{U}(t, y)\Big|_{(1) where\ v\equiv 0} \le \gamma(t) U(t, y)$$

are satisfied.

Then the Nonlinear Non-autonomous system $\dot{y} = g(t, y(t), v(t))$ with $v \equiv 0$ is

(a) Globally asymptotically stable if $\gamma(t)$ is asymptotically stable.
(b) Globally uniformly asymptotically stable if $\gamma(t)$ is uniformly exponentially stable and $\beta_1(t, s)$, $\beta_2(t, s)$ are independent of t .
(c) Globally exponentially stable if $\gamma(t)$ is exponentially stable and there exist $n > 0$ and $p_i \in \mathcal{N}, i = 1,2$ such that $\beta_i(t, s) = p_i(t) S^n, i = 1,2.$
(d) Globally uniformly exponentially stable if $\gamma(t)$ is Uniformly exponentially stable and there exist $n > 0$ and $p_i > 0, i = 1,2$ such that $\beta_i(t, s) = p_i(t) S^n, i = 1,2.$

PROOF:

Let β_1, β_2 be two $\mathcal{NK}_\infty$ functions and a scalar function $\gamma(t) \in \mathbb{PC}(J, R)$, we have

$$\beta_1(t, |y|) \le U(t, y) \le \beta_2(t, |y|) \tag{5}$$

$$\dot{U}(t, y)\Big|_{(1) where\ v\equiv 0} \le \gamma(t) U(t, y) \tag{6}$$

$$\frac{\dot{U}(t,y)}{U(t,y)} \leq \gamma(t), \ \forall t_0 \leq t \in J$$

The inequality (5) becomes

$$\beta_1(t_0,|y|) \leq \beta_1(t,|y|), \qquad \forall t_0 \leq t$$

$$\leq U(t,y(t))$$

$$\beta_1(t_0,|y|) \leq U(t_0,y(t_0))\, \varphi(t,t_0)$$

$$\beta_1(t_0,|y|) \leq \beta_2(t_0,|y(t_0)|)\, \varphi(t,t_0) \qquad (7)$$

Proof of (a):

Let the scalar function $\gamma(t)$ is asymptotically stable.

To prove the Nonlinear Non-autonomous system $\dot{y} = g(t,y(t),v(t))$ is globally asymptotically stable.

Since $\lim\limits_{t\to\infty} \varphi(t,t_0) = 0$.

There exists $T = T(t_0)$, such that $\varphi(t,t_0) \leq 1,\ t \geq t_0 + T(t_0)$.

Assume that,

$$\mu(t_0) \triangleq \max_{s\in[t_0,t_0+T(t_0)]} \{\varphi(t,t_0)\} \geq 1$$

Applying the above in equation (7), which implies that

$$\beta_1(t_0,|y|) \leq \beta_2(t_0,|y(t_0)|)\, \mu(t_0), \qquad \forall t_0 \leq t \in J$$

Given that the function $\beta \in \mathcal{NK}_\infty$ and we take $\beta^{-1}(t,s)$ be the inverse function of $\beta(t,s)$ with respect to the second variable, such that

$$\beta^{-1}(t,\beta(t,s)) \equiv 1$$

For every $\varepsilon > 0$, choose

$$\delta(t_0) = {\beta_2}^{-1}(t_0, \frac{1}{\mu(t_0)} \beta_1(t_0,\varepsilon))$$

$$\delta(t_0)\ \beta_2\ (t_0,\mu(t_0)) = \beta_1(t_0,\varepsilon)$$

$$\beta_2(t_0,\mu(t_0))\ \delta(t_0) = \beta_1(t_0,\varepsilon) \qquad (8)$$

Comparing the equations (7) & (8), which gives

$$|y(t_0)| \leq \delta(t_0) \qquad (9)$$

Apply the equation (9) in (7),

$$\beta_1\ (t_0,|y|) \leq \beta_2(t_0,\delta(t_0))\mu(t_0)$$

$$= \beta_1(t_0,\varepsilon)$$

$$|y(t)| \leq \varepsilon, \quad \forall\ t_0 \leq t$$

$$\lim_{t\to\infty} |y(t)| = 0$$

Therefore the nonlinear system is globally asymptotically stable.

Proof of (b):

Assume that $\gamma(t)$ is uniformly exponentially stable and $\beta_1(t,s)$, $\beta_2(t,s)$ are independent of t.

To prove the system is globally uniformly asymptotically stable.

From equation (7), which implies that

$$|y(t)| \leq {\beta_1}^{-1}(\beta_2(|y(t_0)|)\,\varphi(t,t_0)$$

$$\leq {\beta_1}^{-1}(\beta_2(|y(t_0)|)\,exp(\int_{t_0}^{t}\gamma(s)\,ds)$$

$$\leq {\beta_1}^{-1}(\beta_2(|y(t_0)|)\,exp(-\beta(t-t_0)+\alpha(t_0))$$

$$|y(t)| \leq {\beta_1}^{-1}(\beta_2(|y(t_0)|)\,exp(-\beta(t-t_0))\,exp(\alpha(t_0))$$

Therefore the nonlinear system is globally uniformly asymptotically stable.

Proof of (c):

To prove the system is globally exponentially stable.

Note that,

$${\beta_1}^{-1}(t,s) = s^{1/n}\,{p_i}^{-1/n}(t).$$

From equation (7), which implies that

$$|y(t_0)| \leq {\beta_1}^{-1}(t_0)(\beta_2(t_0,|y(t_0)|)\,\varphi(t,t_0)$$

$$\leq {p_1}^{-1/n}(t_0)(p_2(t_0)(|y(t_0)|^n)\,exp(\int_{t_0}^{t}\gamma(s)\,ds))$$

$$\leq {p_i}^{-1/n}(t_0)(p_2(t_0)(|y(t_0)|^n)\,exp(-\beta(t-t_0)+\alpha(t_0)))^{1/n}$$

$$= [p_2(t_0)/p_1(t_0)]^{1/n}\;1/n\,|y(t_0)|\,exp(-\beta(t-t_0)+\alpha(t_0))$$

$$|y(t_0)| = [p_2(t_0)/p_1(t_0)]^{\frac{1}{n}}\,exp[\alpha(t_0)/n]|y(t_0)|\,exp[-\beta(t-t_0)/n] \qquad (10)$$

Therefore the system is globally exponentially stable.

Proof of (d):

Assume $\gamma(t)$ is uniformly exponentially stable.

To prove the system is globally uniformly exponentially stable.

From equation (10), p_1, p_2 and $\alpha(t_0)$ are independent of t_0. So the nonlinear non-autonomous system $\dot{y} = g(t, y(t), v(t))$ with $v \equiv 0$ is globally uniformly exponentially stable.

THEOREM: 2

Assume that there exist two $\mathcal{NK}_\infty$ functions β_1, β_2 and a C^1 function $U: J \times R^n \to [0,\infty)$ and an asymptotically stable function $\gamma(t) \in \mathbb{PC}(J,R)$ and a scalar function $\chi(t) \in \mathbb{PC}(J,[0,\infty))$ such that for all $(t,y) \in J \times R^n$,

$$\beta_1(t,|y|) \leq U(t,y) \leq \beta_2(t,|y|)$$

$$\dot{U}(t,y)\Big|_{(1)where\ v\equiv 0} \leq \gamma(t)U(t,y) + \chi(t) \qquad (11)$$

are satisfied.

Let $\varphi(t,s) = exp\left(\int_{t_0}^{t}\gamma(s)\,ds\right)$, $t_0 \leq t \in J$ and define $\eta(t,t_0): J \times J \to R$ as,

$$\eta(t, t_0) = \int_{t_0}^{t} \varphi(t, s)\, \chi(s)\, ds.$$

Then the nonlinear non-autonomous system $\dot{y} = g(t, y(t), v(t))$ is globally asymptotically stable if $\eta(t, t_0)$ is bounded for any $t_0 \leq t \in J$ and

$$\lim_{t\to\infty} \eta(t, t_0) = \lim_{t\to\infty} \int_{t_0}^{t} \varphi(t, s)\, \chi(s)\, ds = 0, \quad \forall t_0 \in J.$$

PROOF:

Given that

$$\beta_1(t, |y|) \leq U(t, y) \leq \beta_2(t, |y|)$$

$$\dot{U}(t, y)\Big|_{(1)where\ v\equiv 0} \leq \gamma(t)U(t, y) + \chi(t)$$

$$\beta_1(t_0, |y(t)|) \leq \beta_1(t, |y(t)|), \ \forall t_0 \leq t$$

$$\beta_1(t_0, |y(t)|) \leq U\big(t, y(t)\big) \tag{12}$$

We know the Gronwall –Bellman Inequality, which says that, " Assume that $\gamma(t), \chi(t) \in \mathbb{PC}(J, R)$ and $x(t): J \to [0, \infty)$ be such that

$$\dot{x}(t) \leq \gamma(t)x(t) + \chi(t), \quad \forall t \in J.$$

Then the inequality

$$x(t) \leq x(s)\varphi(t, s) + \int_{s}^{t} \varphi(t, \lambda)\, \chi(\lambda) d\lambda$$

holds true, for any $s \leq t \in J$..

So we have,

$$\beta_1(t_0, |y(t)|) \leq U(t_0, y(t_0))\varphi(t, t_0) + \int_{t_0}^{t} \varphi(t, t_0)\, \chi(s)\, ds$$

$$\beta_1(t_0, |y(t)|) \leq U(t_0, y(t_0))\varphi(t, t_0) + \eta(t, t_0)$$

$$\lim_{t\to\infty} \beta_1(t_0, |y(t)|) \leq \lim_{t\to\infty} U(t_0, y(t_0))\varphi(t, t_0) + \eta(t, t_0)$$

$$\lim_{t\to\infty} \beta_1(t_0, |y(t)|) = 0$$

Since $\lim_{t\to\infty} \varphi(t, t_0) = 0$

Which implies that $\lim_{t\to\infty} |y(t)| = 0$ and $\eta(t, t_0), \varphi(t, t_0)$ are bounded, then $\beta_1(t_0, |y(t)|)$ is bounded.

Therefore $|y(t)|$ is bounded.

Hence the nonlinear non-autonomous system $\dot{y} = g(t, y(t), v(t))$ is globally asymptotically stable.

4. CONCLUSION

In this paper, the stability of non-autonomous nonlinear systems with indefinite derivative of a Lyabunov function are presented. For asymptotic stability, exponential stability theorems can be investigated by introducing scalar function.

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